Symbolic calculus on manifolds with SageMath

Éric Gourgoulhon

Laboratoire Univers et Théories (LUTH)
Observatoire de Paris, CNRS, Université PSL, Université Paris Cité
Meudon, France

and

Laboratoire de Mathématiques de Bretagne Atlantique
CNRS, Université de Bretagne Occidentale
Brest, France

https://luth.obspm.fr/~luthier/gourgoulhon

Albert Einstein Institute

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Outline

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SageMath in a few words

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SageMath is based on Python

- no need to learn any specific syntax to use it
- Python is a powerful *object oriented language*, with a neat syntax
- SageMath benefits from the Python ecosystem (e.g. Jupyter notebook, NumPy, Matplotlib)

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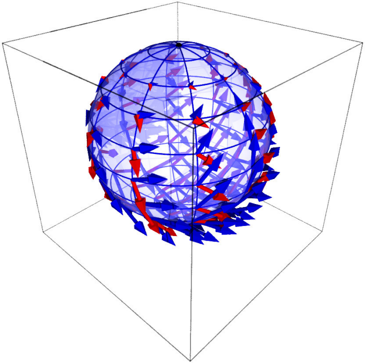
SageMath is developed by an enthusiastic community

- mostly composed of mathematicians
- welcoming newcomers

Differential geometry with SageMath

SageManifolds project: extends SageMath towards **differential geometry** and **tensor calculus**



Stereographic-coordinate frame on $\mathbb{S}^2$

- https://sagemanifolds.obspm.fr
- $\sim$ 119,000 lines of Python code
- fully included in SageMath (after review process)
- $\sim$ 30 contributors (developers and reviewers) cf. https://sagemanifolds.obspm.fr/authors.html
- dedicated mailing list
- help desk: https://ask.sagemath.org

Everybody is welcome to contribute
$\Longrightarrow$ visit https://sagemanifolds.obspm.fr/contrib.html

Current status

Already present (SageMath 10.2):

- differentiable manifolds: tangent spaces, vector frames, tensor fields, curves, pullback and pushforward operators, submanifolds
- vector bundles (tangent bundle, tensor bundles)
- standard tensor calculus (tensor product, contraction, symmetrization, etc.), even on non-parallelizable manifolds, and with all monoterm tensor symmetries taken into account
- Lie derivative along a vector field
- differential forms: exterior and interior products, exterior derivative, Hodge duality
- multivector fields: exterior and interior products, Schouten-Nijenhuis bracket
- affine connections (curvature, torsion)
- pseudo-Riemannian metrics
- computation of geodesics by numerical integration *(thanks to Karim!)*

Current status

Already present *(cont'd)*:

- some plotting capabilities (charts, points, curves, vector fields)
- parallelization (on tensor components) of CPU demanding computations
- extrinsic geometry of pseudo-Riemannian submanifolds
- series expansions of tensor fields
- symplectic manifolds
- 2 symbolic backends: Pynac/Maxima (SageMath's default) and SymPy

Future prospects:

- more symbolic backends (Giac, FriCAS, ...)
- more graphical outputs
- spinors, integrals on submanifolds, variational calculus, etc.
- connection with numerical relativity: use SageMath to explore numerically-generated spacetimes

Outline

1. SageMath and its differential geometry capabilities
2. SageMath implementation of tensor fields
3. Example: Schwarzschild spacetime
4. Other examples
5. Conclusions

Vector fields on a smooth manifold

The set $\mathfrak{X}(M)$ of vector fields on a smooth manifold M over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ is endowed with two algebraic structures:

1. $\mathfrak{X}(M)$ is an infinite-dimensional vector space over $\mathbb{K}$, the scalar multiplication $\mathbb{K} \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(\lambda, \boldsymbol{v}) \mapsto \lambda \boldsymbol{v}$ being defined by

$$\forall p \in M, \quad (\lambda \boldsymbol{v})|_p = \lambda \boldsymbol{v}|_p \,,$$

2. $\mathfrak{X}(M)$ is a module over the commutative algebra $C^\infty(M)$, the scalar multiplication $C^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(f, \boldsymbol{v}) \mapsto f\boldsymbol{v}$ being defined by

$$\forall p \in M, \quad (f\boldsymbol{v})|_p = f(p)\boldsymbol{v}|_p \,,$$

   the right-hand side involving the scalar multiplication by $f(p) \in \mathbb{K}$ in the vector space T_pM.

$\mathfrak{X}(M)$ as a $C^\infty(M)$-module

$\mathfrak{X}(M)$ is a ***free module*** over $C^\infty(M) \iff \mathfrak{X}(M)$ admits a basis

If this occurs, then $\mathfrak{X}(M)$ is actually a ***free module of finite rank*** over $C^\infty(M)$ and $\mathrm{rank}\,\mathfrak{X}(M) = \dim M = n$.
One says then that M is a ***parallelizable*** manifold.
A basis $(\boldsymbol{e}_a)_{1\leq a\leq n}$ of $\mathfrak{X}(M)$ is called a ***vector frame***

Basis expansion[1]:

$$\forall \boldsymbol{v} \in \mathfrak{X}(M), \quad \boldsymbol{v} = v^a \boldsymbol{e}_a, \quad \text{with } v^a \in C^\infty(M) \tag{1}$$

At each point $p \in M$, (1) gives birth to an identity in the tangent space T_pM:

$$\left.\boldsymbol{v}\right|_p = v^a(p)\, \left.\boldsymbol{e}_a\right|_p, \quad \text{with } v^a(p) \in \mathbb{K},$$

which is nothing but the expansion of the tangent vector $\left.\boldsymbol{v}\right|_p$ on the basis $(\left.\boldsymbol{e}_a\right|_p)_{1\leq a\leq n}$ of the vector space T_pM.

[1] Einstein's convention for summation on repeated indices is assumed.

Parallelizable manifolds

M is **parallelizable** $\iff$ $\mathfrak{X}(M)$ is a free $C^\infty(M)$-module of rank n
$\iff$ M admits a global vector frame
$\iff$ the tangent bundle is trivial: $TM \simeq M \times \mathbb{K}^n$

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Examples of parallelizable manifolds

- $\mathbb{R}^n$ (global coordinate chart $\Rightarrow$ global vector frame)
- the circle $\mathbb{S}^1$ (*rem:* no global coordinate chart)
- the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$
- the 3-sphere $\mathbb{S}^3 \simeq \mathrm{SU}(2)$, as any Lie group
- the 7-sphere $\mathbb{S}^7$
- any orientable 3-manifold (Steenrod theorem)

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Examples of non-parallelizable manifolds

- the sphere $\mathbb{S}^2$ (hairy ball theorem!) and any n-sphere $\mathbb{S}^n$ with $n \notin \{1, 3, 7\}$
- the real projective plane $\mathbb{RP}^2$

SageMath implementation of vector fields

Choice of the $C^\infty(M)$-module point of view for $\mathfrak{X}(M)$, instead of the infinite-dimensional $\mathbb{K}$-vector space one

$\Longrightarrow$ implementation advantages:

- reduction to finite-dimensional structures: free $C^\infty(U)$-modules of rank n on parallelizable open subsets $U \subset M$
- for tensor calculus on each parallelizable open set U, use of exactly the same `FiniteRankFreeModule` code as for tangent spaces

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Decomposition of M into parallelizable parts

Assumption: the smooth manifold M can be covered by a finite number m of parallelizable open subsets U_i $(1 \leq i \leq m)$

Example: this holds if M is compact (finite atlas)

More details on the implementation:

[E. Gourgoulhon & M. Mancini, Les cours du CIRM **6**, 1 (2018)]

Tensor field storage

TensorField
T

dictionary TensorField._restrictions

- domain 1: U_1 — TensorFieldParal: $T|_{U_1} = T^a{}_b e_a \otimes e^b = T^{\hat{a}}{}_{\hat{b}} \varepsilon_{\hat{a}} \otimes \varepsilon^{\hat{b}} = \ldots$
- domain 2: U_2 — TensorFieldParal: $T|_{U_2}$
- $\ldots$

dictionary TensorFieldParal._components

- frame 1: (e_a) — Components: $(T^a{}_b)_{1 \le a,\, b \le n}$
- frame 2: $(\varepsilon_{\hat{a}})$ — Components: $(T^{\hat{a}}{}_{\hat{b}})_{1 \le \hat{a},\, \hat{b} \le n}$
- $\ldots$

dictionary Components._comp

- $(1,1)$: DiffScalarField: $T^1{}_1$
- $(1,2)$: DiffScalarField: $T^1{}_2$
- $\ldots$

dictionary DiffScalarField._express

- chart 1: (x^a) — ChartFunction: $T^1{}_1\left(x^1, \ldots, x^n\right)$
- chart 2: (y^a) — ChartFunction: $T^1{}_1\left(y^1, \ldots, y^n\right)$
- $\ldots$

dictionary ChartFunction._express

- SR: Expression: $x^1 \cos x^2$
- SymPy: Basic: $x^1 \cos x^2$
- $\ldots$

Outline

Examples of SageMath computations on Schwarzschild spacetime

SageMath Jupyter notebook:
https://nbviewer.jupyter.org/github/egourgoulhon/SageMathTour/blob/master/Notebooks/demo_Schwarzschild.ipynb

A longer example with computation of geodesics:
https://nbviewer.org/github/egourgoulhon/SageMathTour/blob/master/Notebooks/demo_pseudo_Riemannian_Schwarzschild.ipynb

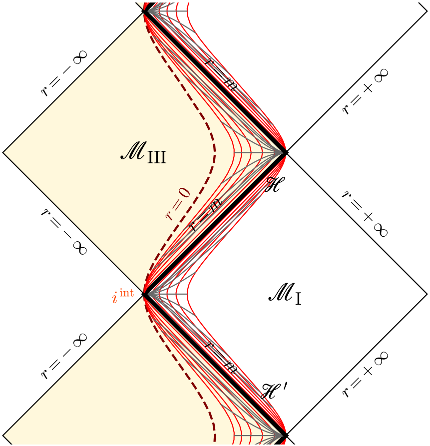
(In the nbviewer menu, click on to run an interactive version on a Binder server)

Outline

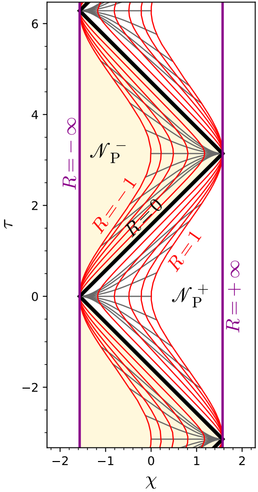
Other examples

- **More on Schwarzschild** (Kruskal-Szekeres and isotropic coordinates):
  https://nbviewer.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_Schwarzschild.ipynb
- **Near-horizon geometry of the extremal Kerr black hole**:
  https://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_extremal_Kerr_near_horizon.ipynb
- **Computation of geodesics in Kerr spacetime**:
  https://nbviewer.jupyter.org/github/BlackHolePerturbationToolkit/kerrgeodesic_gw/blob/master/Notebooks/Kerr_geodesics.ipynb
- **Tolman-Oppenheimer-Volkoff equations** (derivation of TOV system and numerical integration):
  https://nbviewer.org/github/sagemanifolds/SageManifolds/blob/master/Worksheets/v1.3/SM_TOV.ipynb

Carter-Penrose diagrams generated with SageMath



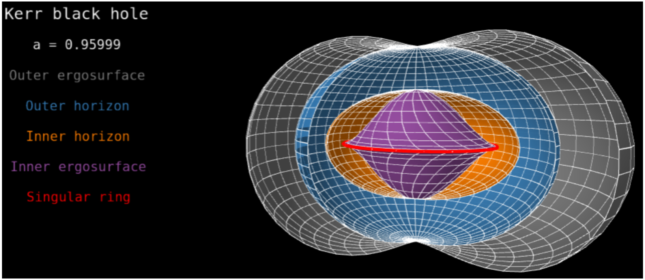
Extremal Kerr



NHEK spacetime

https://nbviewer.org/github/egourgoulhon/BHLectures/blob/master/sage/Kerr_extremal_extended.ipynb
https://nbviewer.org/github/egourgoulhon/BHLectures/blob/master/sage/NHEK_spacetime.ipynb

Animated view of horizons and ergosurfaces in Kerr spacetime



The notebook:
https://nbviewer.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_Kerr_surfaces.ipynb

The animated view:
https://sagemanifolds.obspm.fr/images/animated/Kerr_surfaces.html

Image of an accretion disk surrounding a Schwarzschild BH

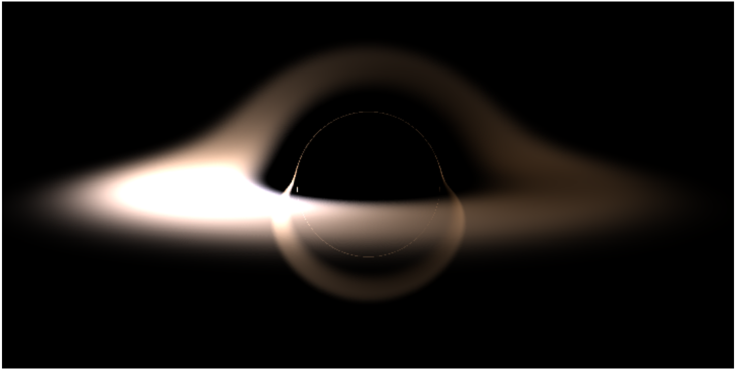


Image computed with SageMath by integrating null geodesics, cf. the notebook
https://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_black_hole_rendering.ipynb

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Many examples available at
`https://sagemanifolds.obspm.fr/examples.html`

Want to join the SageManifolds project or to simply stay tuned?

visit `https://sagemanifolds.obspm.fr/`
(download, documentation, example notebooks, mailing list)