A generalized Damour-Navier-Stokes equation applied to trapping horizons

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Plan

Outline

Concept of black hole viscosity

- Hartle (1973): introduced the concept of **black hole viscosity** when studying the response of the *event horizon* to external perturbations
- Damour (1979): 2-dimensional **Navier-Stokes** like equation for the event horizon $\Longrightarrow$ *shear viscosity* and *bulk viscosity*
- Thorne and Price (1986): **membrane paradigm** for black holes

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Shall we restrict the analysis to the event horizon ?

Event horizon = extremely global and teleological concept

Location of the event horizon requires the knowledge of the full spacetime (in particular of the full future of an initial Cauchy surface)
Not appropriate for 3+1 numerical relativity

Recently local characterization of black hole have been introduced

- Hayward (1994): **future trapping horizon** = hypersurface foliated by marginally trapped 2-surfaces
- Ashtekar, Beetle & Fairhurst (1999): **isolated horizon** = null hypersurface whose intrinsic and extrinsic geometry is not evolving along its null generators
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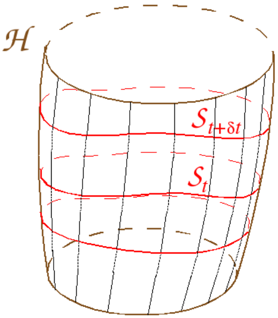
Extend the concept of viscosity to these hypersurfaces ?

NB: *event horizon* = null hypersurface
future trapping horizon = null or spacelike hypersurface

Outline

Foliation of a hypersurface by spacelike 2-surfaces



hypersurface $\mathcal{H}$ = submanifold of spacetime $(\mathcal{M}, \boldsymbol{g})$ of codimension 1

$$\mathcal{H} \text{ can be } \begin{cases} \text{spacelike} \\ \text{null} \\ \text{timelike} \end{cases}$$

$$\mathcal{H} = \bigcup_{t \in \mathbb{R}} \mathcal{S}_t$$

$\mathcal{S}_t$ = spacelike 2-surface

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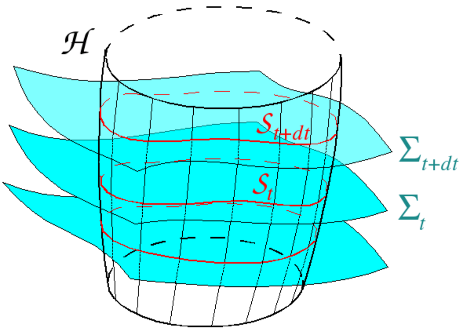

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$\Longleftarrow$ 3+1 perspective

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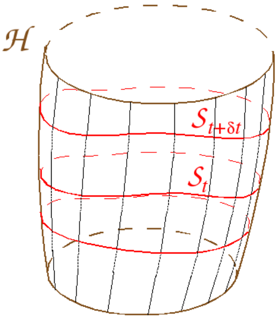

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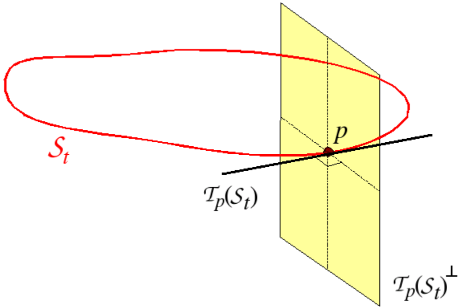
intrinsic viewpoint adopted here (i.e. not relying on extra-structure such as a 3+1 foliation)

Orthogonal projector on $\mathcal{S}_t$

$\mathcal{S}_t$ spacelike $\iff$ induced metric $\boldsymbol{q}$ positive definite
$\boldsymbol{q}$ not degenerate $\Longrightarrow$ orthogonal decomposition of the tangent space at any $p \in \mathcal{M}$:

$$\mathcal{T}_p(\mathcal{M}) = \mathcal{T}_p(\mathcal{S}_t) \oplus \mathcal{T}_p(\mathcal{S}_t)^{\perp}$$



$\boldsymbol{q}$: induced metric on $\mathcal{S}_t$, components: $q_{\alpha\beta}$
$\vec{\boldsymbol{q}}$: orthogonal projector onto $\mathcal{S}_t$, components: $q^{\alpha}{}_{\beta}$

Projection operator $\vec{q}^*$

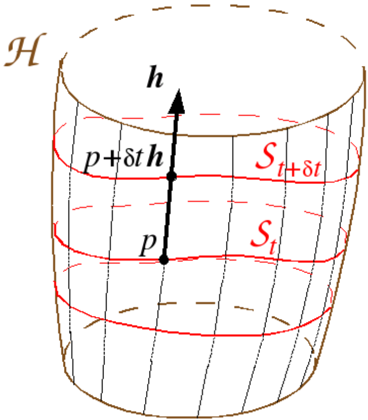
$\boldsymbol{A}$: tensor of covariance type (m, n)
$\vec{\boldsymbol{q}}^* \boldsymbol{A}$: tensor of same covariance type, defined by

$$(\vec{q}^* A)^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n} := q^{\alpha_1}{}_{\mu_1} \cdots q^{\alpha_m}{}_{\mu_m} q^{\nu_1}{}_{\beta_1} \cdots q^{\nu_n}{}_{\beta_n} A^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n}$$

Remark: for a vector: $\vec{\boldsymbol{q}}^* \boldsymbol{v} = \vec{\boldsymbol{q}}(\boldsymbol{v})$
for a 1-form, $\vec{\boldsymbol{q}}^* \boldsymbol{\omega} = \boldsymbol{\omega} \circ \vec{\boldsymbol{q}}$

Definition: a tensor $\boldsymbol{A}$ is *tangent to* $\mathcal{S}_t$ iff $\vec{\boldsymbol{q}}^* \boldsymbol{A} = \boldsymbol{A}$.

Evolution vector



Vector field $\boldsymbol{h}$ on $\mathcal{H}$ defined by

- (i) $\boldsymbol{h}$ is tangent to $\mathcal{H}$
- (ii) $\boldsymbol{h}$ is orthogonal to $\mathcal{S}_t$
- (iii) $\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, t = h^\mu \partial_\mu t = \langle \mathbf{d}t, \boldsymbol{h} \rangle = 1$

NB: (iii) $\Longrightarrow$ the 2-surfaces $\mathcal{S}_t$ are Lie-dragged by $\boldsymbol{h}$

Lie derivatives along $\boldsymbol{h}$

Since the 2-surfaces $\mathcal{S}_t$ are Lie-dragged by $\boldsymbol{h}$, so are their tangent vectors:

$$\forall \boldsymbol{v} \in \mathcal{T}(\mathcal{S}_t),\ \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{v} \in \mathcal{T}(\mathcal{S}_t)$$

i.e. $\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}$ = internal operator on $\mathcal{T}(\mathcal{S}_t)$

Extension to 1-forms in $\mathcal{T}^*(\mathcal{S}_t)$:

$$\forall \boldsymbol{v} \in \mathcal{T}(\mathcal{S}_t), \quad \langle \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{\omega}, \boldsymbol{v}\rangle := \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \langle \boldsymbol{\omega}, \boldsymbol{v}\rangle - \langle \boldsymbol{\omega}, \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{v}\rangle.$$

Extension to any tensor $\boldsymbol{A}$ tangent to $\mathcal{S}_t$ by tensor products

Definition:

$${}^{\mathcal{S}}\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{A} := \vec{\boldsymbol{q}}^{*} \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{A} = \vec{\boldsymbol{q}}^{*} \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \vec{\boldsymbol{q}}^{*} \boldsymbol{A}$$

Norm of $\boldsymbol{h}$ and type of $\mathcal{H}$

Definition: $C := \frac{1}{2}\,\boldsymbol{h}\cdot\boldsymbol{h}$

$$
\begin{array}{lclcl}
\mathcal{H} \text{ is spacelike} & \iff & C > 0 & \iff & \boldsymbol{h} \text{ is spacelike} \\
\mathcal{H} \text{ is null} & \iff & C = 0 & \iff & \boldsymbol{h} \text{ is null} \\
\mathcal{H} \text{ is timelike} & \iff & C < 0 & \iff & \boldsymbol{h} \text{ is timelike.}
\end{array}
$$

Expansion and shear along normal vectors

Let $\boldsymbol{v}$ be a vector field on $\mathcal{H}$ everywhere normal to $\mathcal{S}_t$.

Deformation tensor of $\mathcal{S}_t$ along $\boldsymbol{v}$: $\boldsymbol{\Theta}^{(\boldsymbol{v})} := \vec{\boldsymbol{q}}^* \boldsymbol{\nabla} \underline{\boldsymbol{v}}$ or $\Theta^{(\boldsymbol{v})}_{\alpha\beta} := \nabla_\nu v_\mu \, q^\mu_{\ \alpha} q^\nu_{\ \beta}$

$\boldsymbol{v}$ normal to a 2-surface $(\mathcal{S}_t) \Longrightarrow \boldsymbol{\Theta}^{(\boldsymbol{v})}$ is a symmetric bilinear form

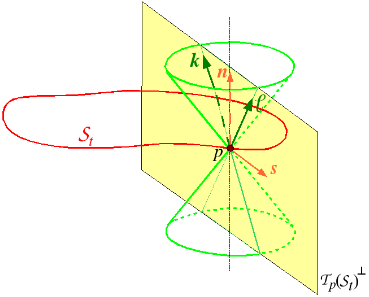
Prop: $\boldsymbol{\Theta}^{(\boldsymbol{v})} = \frac{1}{2} \vec{\boldsymbol{q}}^* \boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \, \boldsymbol{q}$

Decomposition into traceless part (**shear** $\boldsymbol{\sigma}^{(\boldsymbol{v})}$) and trace part (**expansion** $\theta^{(\boldsymbol{v})}$):

$\boldsymbol{\Theta}^{(\boldsymbol{v})} = \boldsymbol{\sigma}^{(\boldsymbol{v})} + \frac{1}{2} \theta^{(\boldsymbol{v})} \, \boldsymbol{q}$ with $\theta^{(\boldsymbol{v})} := q^{\mu\nu} \Theta^{(\boldsymbol{v})}_{\mu\nu} = \boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \ln \sqrt{q}$, $q := \det q_{ab}$

Prop: $\boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \, {}^{\mathcal{S}}\!\boldsymbol{\epsilon} = \theta^{(\boldsymbol{v})} \, {}^{\mathcal{S}}\!\boldsymbol{\epsilon}$ with ${}^{\mathcal{S}}\!\boldsymbol{\epsilon}$ surface element of $(\mathcal{S}_t, \boldsymbol{q})$: ${}^{\mathcal{S}}\!\boldsymbol{\epsilon} = \sqrt{q} \, \mathbf{d}x^2 \wedge \mathbf{d}x^3$

Frames normal to $\mathcal{S}_t$



Two natural types of choice for a vector basis of $\mathcal{T}_p(\mathcal{S}_t)^\perp$:

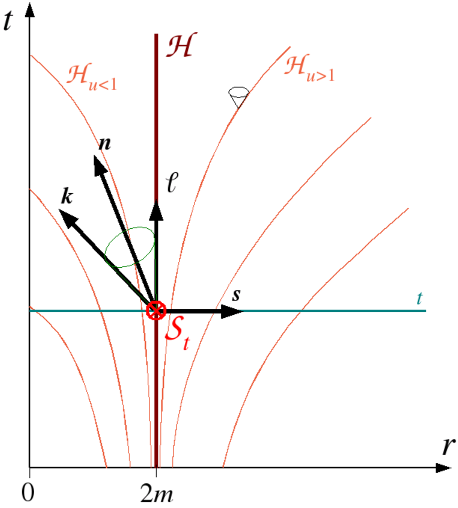
1. an orthonormal basis $(\boldsymbol{n}, \boldsymbol{s})$ ($\boldsymbol{n}$ = timelike, $\boldsymbol{s}$ = spacelike):
   $\boldsymbol{n}\cdot\boldsymbol{n} = -1, \quad \boldsymbol{s}\cdot\boldsymbol{s} = 1, \quad \boldsymbol{n}\cdot\boldsymbol{s} = 0$
2. a pair linearly independent future-directed null vectors $(\boldsymbol{\ell}, \boldsymbol{k})$:
   $\boldsymbol{\ell}\cdot\boldsymbol{\ell} = 0, \quad \boldsymbol{k}\cdot\boldsymbol{k} = 0, \quad \boldsymbol{\ell}\cdot\boldsymbol{k} =: -e^{\sigma}$

Degrees of freedom:

1. boost : $\begin{cases} \boldsymbol{n}' = \cosh\eta\,\boldsymbol{n} + \sinh\eta\,\boldsymbol{s} \\ \boldsymbol{s}' = \sinh\eta\,\boldsymbol{n} + \cosh\eta\,\boldsymbol{s} \end{cases}, \quad \eta \in \mathbb{R}$
2. rescaling : $\begin{cases} \boldsymbol{\ell}' = \lambda\,\boldsymbol{\ell}, & \lambda > 0 \\ \boldsymbol{k}' = \mu\,\boldsymbol{k}, & \mu > 0 \end{cases}$

Orthogonal projector: $\vec{\boldsymbol{q}} = \mathbf{1} + \langle \underline{\boldsymbol{n}}, .\rangle\,\boldsymbol{n} - \langle \underline{\boldsymbol{s}}, .\rangle\,\boldsymbol{s} = \mathbf{1} + e^{-\sigma}\langle \underline{\boldsymbol{k}}, .\rangle\,\boldsymbol{\ell} + e^{-\sigma}\langle \underline{\boldsymbol{\ell}}, .\rangle\,\boldsymbol{k}$

Example of normal frames



$\mathcal{H}$ = event horizon of Schwarzschild black hole
$\mathcal{S}_t$ = slice of constant Eddington-Finkelstein time

Second fundamental tensor of $\mathcal{S}_t$

Tensor $\boldsymbol{\mathcal{K}}$ of type $(1,2)$ relating the covariant derivative of a vector tangent to $\mathcal{S}_t$ taken by the spacetime connection $\boldsymbol{\nabla}$ to that taken by the connection $\boldsymbol{\mathcal{D}}$ in $\mathcal{S}_t$ compatible with the induced metric $\boldsymbol{q}$:

$$\forall(\boldsymbol{u},\boldsymbol{v}) \in \mathcal{T}(\mathcal{S}_t)^2, \quad \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} = \boldsymbol{\mathcal{D}}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{\mathcal{K}}(\boldsymbol{u},\boldsymbol{v})$$

Prop:

$\mathcal{K}^{\alpha}{}_{\beta\gamma} = \nabla_{\mu} q^{\alpha}{}_{\nu}\, q^{\mu}{}_{\beta} q^{\nu}{}_{\gamma}$

$\mathcal{K}^{\alpha}{}_{\beta\gamma} = n^{\alpha}\,\Theta^{(\boldsymbol{n})}_{\beta\gamma} - s^{\alpha}\,\Theta^{(\boldsymbol{s})}_{\beta\gamma} = e^{-\sigma}\left(k^{\alpha}\,\Theta^{(\boldsymbol{\ell})}_{\beta\gamma} + \ell^{\alpha}\,\Theta^{(\boldsymbol{k})}_{\beta\gamma}\right)$

Remark: for a hypersurface of normal $\boldsymbol{n}$ and extrinsic curvature $\boldsymbol{K}$, $\mathcal{K}^{\alpha}{}_{\beta\gamma} = -n^{\alpha} K_{\beta\gamma}$

Normal fundamental forms

Extrinsic geometry of $\mathcal{S}_t$ not entirely specified by $\mathcal{K}$ (contrary to the hypersurface case)

$\mathcal{K}$ involves only the deformation tensors $\boldsymbol{\Theta}^{(\cdot)}$ of the normals to $\mathcal{S}_t \Longrightarrow \mathcal{K}$ encodes only the part of the variation of $\mathcal{S}_t$'s normals which is parallel to $\mathcal{S}_t$

Variation of the two normals with respect to each other: encoded by the **normal fundamental forms** (also called *external rotation coefficients* or *connection on the normal bundle*, or if $\mathcal{H}$ is null, *Hájíček 1-form*):

1. $\boldsymbol{\Omega}^{(\boldsymbol{n})} := \boldsymbol{s} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{n}$ or $\Omega^{(\boldsymbol{n})}_{\alpha} := s_\mu \nabla_\nu n^\mu \, q^\nu{}_\alpha$
   $\boldsymbol{\Omega}^{(\boldsymbol{s})} := \boldsymbol{n} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{s}$
2. $\boldsymbol{\Omega}^{(\boldsymbol{\ell})} := \dfrac{1}{\boldsymbol{k}\cdot\boldsymbol{\ell}}\, \boldsymbol{k} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{\ell}$ or $\Omega^{(\boldsymbol{\ell})}_{\alpha} := \dfrac{1}{k_\rho \ell^\rho} k_\mu \nabla_\nu \ell^\mu \, q^\nu{}_\alpha$
   $\boldsymbol{\Omega}^{(\boldsymbol{k})} := \dfrac{1}{\boldsymbol{k}\cdot\boldsymbol{\ell}}\, \boldsymbol{\ell} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{k}$

Basic properties of the normal fundamental forms

From the definition: $\boldsymbol{\Omega}^{(s)} = -\boldsymbol{\Omega}^{(n)}$ and $\boldsymbol{\Omega}^{(k)} = -\boldsymbol{\Omega}^{(\ell)} + \mathcal{D}\sigma$

Relation between the $(\boldsymbol{n}, \boldsymbol{s})$-type and the $(\boldsymbol{\ell}, \boldsymbol{k})$-type:
$\boldsymbol{\Omega}^{(\ell)} = \boldsymbol{\Omega}^{(n)}$ $[\boldsymbol{\ell} = \boldsymbol{n} + \boldsymbol{s}]$ and $\boldsymbol{\Omega}^{(k)} = -\boldsymbol{\Omega}^{(n)}$ $[\boldsymbol{k} = \boldsymbol{n} - \boldsymbol{s}]$

The normal fundamental forms are not unique

(contrary to the second fundamental tensor $\mathcal{K}$)
Dependence of the normal frame

1. $(\boldsymbol{n}, \boldsymbol{s}) \mapsto (\boldsymbol{n}', \boldsymbol{s}') \Longrightarrow$ $\boxed{\boldsymbol{\Omega}^{(n')} = \boldsymbol{\Omega}^{(n)} + \mathcal{D}\eta}$
2. $(\boldsymbol{\ell}, \boldsymbol{k}) \mapsto (\boldsymbol{\ell}', \boldsymbol{k}') \Longrightarrow$ $\boxed{\boldsymbol{\Omega}^{(\ell')} = \boldsymbol{\Omega}^{(\ell)} + \mathcal{D} \ln \lambda}$

"Surface-gravity" 1-forms

If the vector fields $(\boldsymbol{\ell}, \boldsymbol{k})$ are extended away from $\mathcal{S}_t$, define the 1-form

$$\boxed{\boldsymbol{\kappa}^{(\boldsymbol{\ell})} := \frac{1}{\boldsymbol{k}\cdot\boldsymbol{\ell}}\,\boldsymbol{k}\cdot\boldsymbol{\nabla}_{\boldsymbol{p}}\,\boldsymbol{\ell}}$$ or $$\kappa^{(\boldsymbol{\ell})}_{\alpha} := \frac{1}{k_\rho \ell^\rho} k_\mu \nabla_\nu \ell^\mu \, p^\nu{}_\alpha$$

where $\boldsymbol{p}$ is the orthogonal projector complementary to $\vec{\boldsymbol{q}}$: $\mathbf{1} = \vec{\boldsymbol{q}} + \boldsymbol{p}$.

NB: Since $\boldsymbol{p}$ is a projector in a direction transverse to $\mathcal{S}_t$, the 1-form $\boldsymbol{\kappa}^{(\boldsymbol{\ell})}$ is not intrinsic to the 2-surface $\mathcal{S}_t$: it depends on the choice of $\boldsymbol{\ell}$ away from $\mathcal{S}_t$

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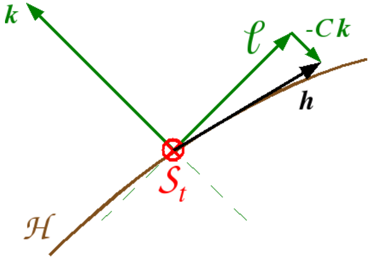
If $\boldsymbol{\ell}$ is extended along one of the two families of light rays emanating radially from $\mathcal{S}_t$, then $\boldsymbol{\ell}$ is pre-geodesic: $\boldsymbol{\nabla}_{\boldsymbol{\ell}}\,\boldsymbol{\ell} = \nu_{(\boldsymbol{\ell})}\,\boldsymbol{\ell}$, with the *inaffinity parameter* (*surface gravity* if $\boldsymbol{\ell} =$ null Killing vector of Kerr spacetime) given by the 1-form $\boldsymbol{\kappa}^{(\boldsymbol{\ell})}$ applied to $\boldsymbol{\ell}$:

$$\nu_{(\boldsymbol{\ell})} = \langle \boldsymbol{\kappa}^{(\boldsymbol{\ell})}, \boldsymbol{\ell} \rangle$$

Normal null frame associated with the evolution vector



The foliation $(\mathcal{S}_t)_{t\in\mathbb{R}}$ entirely fixes the ambiguities in the choice of the null normal frame $(\boldsymbol{\ell}, \boldsymbol{k})$, via the evolution vector $\boldsymbol{h}$: there exists a unique normal null frame $(\boldsymbol{\ell}, \boldsymbol{k})$ such that

$$\boldsymbol{h} = \boldsymbol{\ell} - C\boldsymbol{k} \quad \text{and} \quad \boldsymbol{\ell}\cdot\boldsymbol{k} = -1$$

Outline

Original Damour-Navier-Stokes equation

Hyp: $\mathcal{H}$ = null hypersurface (particular case: black hole **event horizon**)
Then $\boldsymbol{h} = \boldsymbol{\ell}$ ($C = 0$) reminder

Damour (1979) has derived from Einstein equation the relation

$$^{\mathcal{S}}\!\mathcal{L}_{\boldsymbol{\ell}}\,\boldsymbol{\Omega}^{(\ell)} + \theta^{(\ell)}\boldsymbol{\Omega}^{(\ell)} = \boldsymbol{\mathcal{D}}\nu_{(\ell)} - \boldsymbol{\mathcal{D}}\cdot\vec{\boldsymbol{\sigma}}^{(\ell)} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\ell)} + 8\pi\vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}$$

or equivalently

$$^{\mathcal{S}}\!\mathcal{L}_{\boldsymbol{\ell}}\,\boldsymbol{\pi} + \theta^{(\ell)}\boldsymbol{\pi} = -\boldsymbol{\mathcal{D}}P + 2\eta\boldsymbol{\mathcal{D}}\cdot\vec{\boldsymbol{\sigma}}^{(\ell)} + \xi\boldsymbol{\mathcal{D}}\theta^{(\ell)} + \boldsymbol{f}$$

with

$\boldsymbol{\pi} := -\frac{1}{8\pi}\boldsymbol{\Omega}^{(\ell)}$ momentum surface density

$P := \frac{\nu_{(\ell)}}{8\pi}$ pressure

$\eta := \frac{1}{16\pi}$ shear viscosity

$\xi := -\frac{1}{16\pi}$ bulk viscosity

$\boldsymbol{f} := -\vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}$ external force surface density ($\boldsymbol{T}$ = stress-energy tensor)

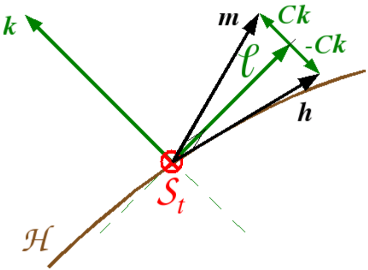
Generalization to the non-null case

Starting remark: in the null case, ℓ plays two different roles:

- evolution vector along $\mathcal{H}$ (e.g. term ${}^{\mathcal{S}}\!\mathcal{L}_{\ell}$)
- normal to $\mathcal{H}$ (e.g. term $\vec{q}^{\,*}\boldsymbol{T}\cdot\boldsymbol{\ell}$)

When $\mathcal{H}$ is no longer null, these two roles have to be taken by two different vectors:

- evolution vector: obviously $\boldsymbol{h}$ reminder
- vector normal to $\mathcal{H}$: a natural choice is $\boxed{\boldsymbol{m} := \boldsymbol{\ell} + C\boldsymbol{k}}$



Generalized Damour-Navier-Stokes equation

Starting point of the calculation: contracted Ricci identity applied to the vector $\boldsymbol{m}$ and projected onto $\mathcal{S}_t$:

$$\left(\nabla_\mu \nabla_\nu m^\mu - \nabla_\nu \nabla_\mu m^\mu\right) q^\nu{}_\alpha = R_{\mu\nu} m^\mu q^\nu{}_\alpha$$

Final result:

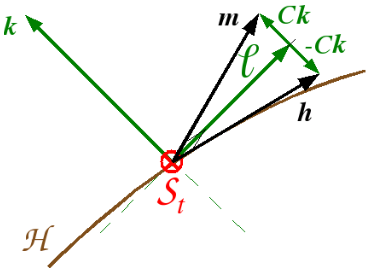
$$^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \theta^{(\boldsymbol{h})}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell})} = \boldsymbol{\mathcal{D}}\langle \boldsymbol{\kappa}^{(\boldsymbol{\ell})}, \boldsymbol{h}\rangle - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(\boldsymbol{m})} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{m})} - \theta^{(\boldsymbol{k})}\boldsymbol{\mathcal{D}}C + 8\pi \vec{\boldsymbol{q}}^{*}\boldsymbol{T} \cdot \boldsymbol{m}$$

- $\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$: normal fundamental form of $\mathcal{S}_t$ associated with null normal $\boldsymbol{\ell}$ reminder
- $\theta^{(\boldsymbol{h})}$, $\theta^{(\boldsymbol{m})}$ and $\theta^{(\boldsymbol{k})}$: expansion scalars of $\mathcal{S}_t$ along the vectors $\boldsymbol{h}$, $\boldsymbol{m}$ and $\boldsymbol{k}$ respectively reminder
- $\boldsymbol{\mathcal{D}}$: covariant derivative within $(\mathcal{S}_t, \boldsymbol{q})$
- $\boldsymbol{\kappa}^{(\boldsymbol{\ell})}$: "surface-gravity" 1-form associated with the null vector $\boldsymbol{\ell}$ reminder
- $\boldsymbol{\sigma}^{(\boldsymbol{m})}$: shear tensor of $\mathcal{S}_t$ along the vector $\boldsymbol{m}$ reminder
- C : half the scalar square of $\boldsymbol{h}$ reminder

Null limit

In the null limit,

$$\boldsymbol{h} = \boldsymbol{m} = \boldsymbol{\ell} \qquad \text{and} \qquad C = 0$$



and we recover the original Damour-Navier-Stokes equation:

$$^{\mathcal{S}}\mathcal{L}_{\ell}\,\boldsymbol{\Omega}^{(\ell)} + \theta^{(\ell)}\boldsymbol{\Omega}^{(\ell)} = \boldsymbol{\mathcal{D}}\nu_{(\ell)} - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(\ell)} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\ell)} + 8\pi \vec{\boldsymbol{q}}^{*}\boldsymbol{T} \cdot \boldsymbol{\ell}$$

Behavior under a change of normal fundamental form

$\boldsymbol{\ell} \mapsto \boldsymbol{\ell}' = \lambda \boldsymbol{\ell} \Longrightarrow \boldsymbol{\Omega}^{(\ell')} = \boldsymbol{\Omega}^{(\ell)} + \boldsymbol{\mathcal{D}} \ln \lambda$ and $\boldsymbol{\kappa}^{(\ell')} = \boldsymbol{\kappa}^{(\ell)} + \boldsymbol{\nabla}_{\boldsymbol{p}} \ln \lambda$

$\Longrightarrow$ generalized Damour-Navier-Stokes equation:

$$\begin{aligned} {}^{\mathcal{S}}\!\mathcal{L}_{\boldsymbol{h}}\, \boldsymbol{\Omega}^{(\ell')} + \theta^{(h)}\, \boldsymbol{\Omega}^{(\ell')} &= \boldsymbol{\mathcal{D}} \langle \boldsymbol{\kappa}^{(\ell')}, \boldsymbol{h} \rangle - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(m)} + \frac{1}{2} \boldsymbol{\mathcal{D}} \theta^{(m)} + \theta^{(\ell)} \boldsymbol{\mathcal{D}} \ln \lambda \\ &\quad - \theta^{(k)} \left(\boldsymbol{\mathcal{D}} C + C \boldsymbol{\mathcal{D}} \ln \lambda \right) + 8\pi \vec{\boldsymbol{q}}^{*} \boldsymbol{T} \cdot \boldsymbol{m} \end{aligned}$$

Choice: $\boldsymbol{\ell}' = \tilde{\boldsymbol{\ell}} =$ null *geodesic* vector along the light rays emanating radially from $\mathcal{S}_t$ ($\mathbf{d}\underline{\tilde{\ell}} = 0$), then $\boldsymbol{\mathcal{D}} C + C \boldsymbol{\mathcal{D}} \ln \lambda = 0$ and the equation reduces to

$${}^{\mathcal{S}}\!\mathcal{L}_{\boldsymbol{h}}\, \boldsymbol{\Omega}^{(\tilde{\ell})} + \theta^{(h)}\, \boldsymbol{\Omega}^{(\tilde{\ell})} = \boldsymbol{\mathcal{D}} \langle \boldsymbol{\kappa}^{(\tilde{\ell})}, \boldsymbol{h} \rangle - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(m)} + \frac{1}{2} \boldsymbol{\mathcal{D}} \theta^{(m)} + \theta^{(\ell)} \boldsymbol{\mathcal{D}} \ln \lambda + 8\pi \vec{\boldsymbol{q}}^{*} \boldsymbol{T} \cdot \boldsymbol{m}$$

Application to future trapping horizons

Definition (Hayward 1994) : $\mathcal{H}$ is a **future trapping horizon** iff $\theta^{(\boldsymbol{\ell})} = 0$ and $\theta^{(\boldsymbol{k})} < 0$.

The generalized Damour-Navier-Stokes equation reduces then to

$$ {}^{\mathcal{S}}\!\mathcal{L}_{\boldsymbol{h}}\, \boldsymbol{\Omega}^{(\tilde{\ell})} + \theta^{(\boldsymbol{h})}\, \boldsymbol{\Omega}^{(\tilde{\ell})} = \boldsymbol{\mathcal{D}}\langle \boldsymbol{\kappa}^{(\tilde{\ell})}, \boldsymbol{h}\rangle - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(\boldsymbol{m})} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{m})} + 8\pi \vec{\boldsymbol{q}}^{*}\boldsymbol{T} \cdot \boldsymbol{m} $$

NB: It has exactly the same structure than Damour's original equation reminder: apart from substitutions of $\boldsymbol{\ell}$ by either $\boldsymbol{h}$ or $\boldsymbol{m}$, it does not contain any extra term

Outline

Generalized angular momentum

Definition [Booth & Fairhurst, gr-qc/0505049)]: Let φ be a vector field on $\mathcal{H}$ which

- is tangent to $\mathcal{S}_t$
- has closed orbits
- has vanishing divergence with respect to the induced metric: $\mathcal{D} \cdot \varphi = 0$

The *generalized angular momentum associated with* φ is then defined by

$$J(\varphi) := -\frac{1}{8\pi} \oint_{\mathcal{S}_t} \langle \mathbf{\Omega}^{(\ell)}, \varphi \rangle \, {}^{\mathcal{S}}\epsilon,$$

Remark 1: does not depend upon the choice of null vector ℓ, thanks to the divergence-free property of φ

Remark 2:

- coincides with Ashtekar & Krishnan's definition for a dynamical horizon
- coincides with Brown-York angular momentum if $\mathcal{H}$ is timelike and φ a Killing vector

Angular momentum flux law

Under the supplementary hypothesis that φ is transported along the evolution vector $\boldsymbol{h}$: $\mathcal{L}_{\boldsymbol{h}}\,\boldsymbol{\varphi} = 0$, the generalized Damour-Navier-Stokes equation leads to

$$\frac{d}{dt}J(\boldsymbol{\varphi}) = -\oint_{\mathcal{S}_t} \boldsymbol{T}(\boldsymbol{m},\boldsymbol{\varphi})\,{}^{\mathcal{S}}\boldsymbol{\epsilon} - \frac{1}{16\pi}\oint_{\mathcal{S}_t}\left[\vec{\boldsymbol{\sigma}}^{(\boldsymbol{m})} : \mathcal{L}_{\boldsymbol{\varphi}}\,\boldsymbol{q} - 2\theta^{(\boldsymbol{k})}\boldsymbol{\varphi}\cdot\boldsymbol{\mathcal{D}}C\right]{}^{\mathcal{S}}\boldsymbol{\epsilon}$$

- $\mathcal{H}$ = null hypersurface : $C = 0$ and $\boldsymbol{m} = \boldsymbol{\ell}$:

$$\frac{d}{dt}J(\boldsymbol{\varphi}) = -\oint_{\mathcal{S}_t} \boldsymbol{T}(\boldsymbol{\ell},\boldsymbol{\varphi})\,{}^{\mathcal{S}}\boldsymbol{\epsilon} - \frac{1}{16\pi}\oint_{\mathcal{S}_t}\vec{\boldsymbol{\sigma}}^{(\boldsymbol{\ell})} : \mathcal{L}_{\boldsymbol{\varphi}}\,\boldsymbol{q}\;{}^{\mathcal{S}}\boldsymbol{\epsilon}$$

  i.e. Eq. (6.134) of the *Membrane Paradigm* book (Thorne, Price & MacDonald 1986)

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Two interesting limiting cases:

- $\mathcal{H}$ = null hypersurface : $C = 0$ and $\boldsymbol{m} = \boldsymbol{\ell}$:

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