Foundation of quantum mechanics and gauge field theories in scale relativity*

Laurent Nottale
CNRS, LUTH, Observatoire de Paris-Meudon,
5 place Jules Janssen, 92195 Meudon Cedex, France
e-mail: laurent.nottale@obspm.fr

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Abstract

We have suggested to extend the principle of relativity to scale transformations of the reference systems, then to apply this "principle of scale relativity" to a description of the space-time geometry generalized in terms of a nondifferentiable (therefore fractal) continuum. In such a framework, one can derive from first principles the main postulates of quantum mechanics. The basic tools of quantum mechanics (complex and spinor wave functions) are constructed as consequences of the nondifferentiable geometry, and the equations they satisfy (Schrödinger, Klein-Gordon, Pauli and Dirac equations) are derived as integrals of the equations of space-time geodesics. Moreover, a new geometric interpretation of the gauge fields (Abelian and non-Abelian) may be proposed in this theory. In this approach, the internal resolution variables (which form a "scale space") become functions of the space-time coordinates. Their transformations can then be identified with the usual gauge transformations. The gauge fields naturally emerge as a manifestation of the fractal geometry, and the gauge charges as the conservative quantities which are built from the symmetries of the scale space.

1 Introduction

One of the main open questions of modern physics is that of the foundation from first principles of quantum physics. The theory of scale relativity provides us with an extension, both of the foundation of physical theories and of the principle of relativity. So it is worth asking such a question in its frame of thought. In scale relativity, one extends the founding stones of physics by giving up the hypothesis of space-time differentiability. Then one proves that a nondifferentiable continuum is fractal, i.e., explicitly dependent on the scales of resolution. This leads one to extend the principle of relativity by applying it, not only to motion laws, but also to the laws of scale transformation of these internal resolution variables.

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In the present paper, we summarize the steps by which one recovers, in this framework, the main postulates of quantum mechanics and of gauge field theories. A more detailed account can be found in Refs. [9, 17, 22, 23].

2 Derivation of the equations of quantum mechanics in scale relativity

2.1 Foundations of scale relativity

The theory of scale relativity is based on the giving up of the hypothesis of manifold differentiability. In this theory, the coordinate transformations are continuous but can be nondifferentiable. The giving up of the assumption of differentiability implies several consequences [1], leading to the following steps of construction.

(1) It has been proved [1, 2, 3] that a continuous and nondifferentiable curve is fractal in a general meaning, namely, its length is explicitly scale-dependent, $\mathcal{L} = \mathcal{L}(\varepsilon)$, and $\mathcal{L} \to \infty$ when $\varepsilon \to 0$. This theorem can be readily extended to a continuous and nondifferentiable manifold.

(2) The fractality of space-time [1, 4, 5, 6] involves the scale dependence of the reference frames. One therefore adds to the usual variables defining the coordinate system, new variables ε characterizing its 'state of scale'. In particular, the coordinates themselves become functions of these scale variables, i.e., $X = X(\varepsilon)$.

(3) The scale variables ε can never be defined in an absolute way, but only in a relative way. Namely, only their ratio $\rho = \varepsilon'/\varepsilon$ does have a physical meaning. This universal behavior leads to extend to scales the principle of relativity [1, 6, 7].

(4) Though non-differentiability manifests itself at the limit $\varepsilon \to 0$, the use of differential equations is made possible by defining fractal functions $f[X(\varepsilon), \varepsilon]$ [1]. While the function $f(X, 0)$ is nondifferentiable, the function $f(X, \varepsilon)$ is differentiable for any $\varepsilon \neq 0$ with respect to both X and ε. We may then use a double differential calculus, in position space and in scale space [7, 2, 8].

(5) The simplest possible scale differential equation is a first order equation, $\partial X/\partial \ln \varepsilon = \beta(X)$, which can be simplified again by Taylor-expanding the unknown function β, so that it reads $\partial X/\partial \ln \varepsilon = a + bX + \cdots$. It is solved as the sum of two terms, a scale-independent, differentiable, 'classical part' and a power-law divergent, scale-dependent, nondifferentiable 'fractal part' [9],

$$X = x + \zeta \left(\frac{\lambda}{\varepsilon} \right)^{-b}, \tag{1}$$

where $x = -a/b$. When b is constant, the second term is the standard expression for the length of a fractal curve of dimension $D_F = 1 - b$ [10]. Moreover, the laws of transformation of this expression under a scale transformation $\ln(\lambda/\varepsilon) \to \ln(\lambda/\varepsilon')$ take the mathematical form of the Galileo group of transformation, and they therefore come, as required, under the principle of relativity [7].

(6) In what follows, we simplify again the description by considering only the case $D_F = 2$, by basing ourselves on Feynman's result [11], according to which the typical paths of quantum particles (those which contribute mainly to the path integral) are nondifferentiable and, in modern words, of fractal dimension $D_F = 2$. The case $D_F \neq 2$ has also been studied in detail (see [2] and references therein). Equation (1) reads, after differentiation and reintroduction of the indices,

$$dX^\mu = dx^\mu + d\xi^\mu = v^\mu ds + \zeta^\mu \sqrt{\lambda_c \, ds}, \tag{2}$$

where λ_c is a length scale which must be introduced for dimensional reasons and which generalizes the Compton length. The ζ^μ are dimensionless highly fluctuating functions. Due to their highly erratic character, we can replace them by stochastic variables such that $<\zeta^\mu>=0$, $<(\zeta^0)^2>=-1$ and $<(\zeta^k)^2>=1$ (k =1 to 3).

(7) Now we can also write the fractal fluctuations in terms of the coordinate differentials, $d\xi^\mu = \zeta^\mu \sqrt{\lambda^\mu \, dx^\mu}$. The identification of this expression with that of Eq. (2) leads to recover the Einstein-de Broglie length and time scales,

$$\lambda_x = \frac{\lambda_c}{dx/ds} = \frac{\hbar}{p_x}, \quad \tau = \frac{\lambda_c}{dt/ds} = \frac{\hbar}{E}. \tag{3}$$

Let us now assume that the large scale (classical) behavior is given by Riemannian metric potentials $g_{\mu\nu}(x, y, z, t)$. The invariant proper time dS along a geodesic writes, in terms of the complete differential elements $dX^\mu = dx^\mu + d\xi^\mu$,

$$dS^2 = g_{\mu\nu} dX^\mu dX^\nu = g_{\mu\nu}(dx^\mu + d\xi^\mu)(dx^\nu + d\xi^\nu). \tag{4}$$

Now replacing the $d\xi$'s by their expression, we obtain a fractal metric [1, 17]. Let us give its two-dimensional and diagonal expression, neglecting the terms of zero mean (in order to simplify its writing):

$$dS^2 = g_{00}(x,t)\left(1 + \zeta_0^2 \, \frac{\tau}{dt}\right) c^2 dt^2 - g_{11}(x,t)\left(1 + \zeta_1^2 \, \frac{\lambda_x}{dx}\right) dx^2. \tag{5}$$

We therefore obtain generalized fractal metric potentials which are divergent and explicitly dependent on the coordinate differential elements, in agreement with Refs. [1, 6]. Another equivalent way to understand this metric consists in remarking that it is no longer only quadratic in the space-time differental elements, but that it also contains them in a linear way.

As a consequence, the curvature is also explicitly scale-dependent and divergent when the scale intervals tend to zero. This property ensures the fundamentally non-Riemannian character of a fractal space-time, as well as the possibility to characterize it in an intrinsic way. Indeed, such a characterization, which is a necessary condition for defining a space in a genuine way, can be easily made by measuring the curvature at smaller and smaller scales. While the curvature vanishes by definition toward the small scales in Gauss-Riemann geometry, a fractal space can be characterized from the interior by the verification of the divergence toward small scales of curvature, and therefore of physical quantities like energy and momentum. Now the expression of this divergence is nothing but the Heisenberg relations themselves, which acquire in this framework the status of a fundamental geometric test of the fractality of space-time [1, 5, 6].

2.2 Geodesics of a fractal space-time

The next step in such a geometric approach consists in the identification of wave-particles with fractal space-time geodesics. Any measurement is interpreted as a selection of the geodesics bundle linked to the interaction with the measurement apparatus (that depends on its resolution) and/or to the information known about it (for example, the which-way-information in a two-slit experiment [2]).

The three main consequences of nondifferentiability are:

(i) The number of fractal geodesics is infinite. We are therefore led to adopt a generalized statistical fluid-like description where the velocity $V^\mu(s)$ is replaced by a scale-dependent velocity field $V^\mu[X^\mu(s, ds), s, ds]$.

(ii) There is a breaking of the reflexion invariance of the differential element ds. Indeed, in terms of fractal functions $f(s, ds)$, two derivatives are defined,

$$X'_+(s, ds) = \frac{X(s + ds, ds) - X(s, ds)}{ds}, \quad X'_-(s, ds) = \frac{X(s, ds) - X(s - ds, ds)}{ds}, \tag{6}$$

which transform one in the other under the reflection ($ds \leftrightarrow -ds$), and which have a priori no reason to be equal. This leads to a fundamental two-valuedness of the velocity field.

(iii) The geodesics are themselves fractal, with fractal dimension $D_F = 2$ playing the role of a critical dimension [2, 11].

This means that one defines two divergent fractal velocity fields, $V_+[x(s, ds), s, ds] = v_+[x(s), s] + w_+[x(s, ds), s, ds]$ and $V_-[x(s, ds), s, ds] = v_-[x(s), s] + w_-[x(s, ds), s, ds]$, which can be decomposed in terms of classical parts v_+ and v_-, and of fractal parts w_+ and w_-.

More generally, we define two "classical" derivatives d_+/ds and d_-/ds, which, when they are applied to x^μ, yield the classical parts of the velocity fields, $v_+^\mu = d_+ x^\mu / ds$ and $v_-^\mu = d_- x^\mu / ds$.

2.3 The Schrödinger equation as a geodesics equation in a fractal space

Let us first consider the non-relativistic case (three-dimensional fractal space, without fractal time), in which the invariant ds is identified with the time differential dt. One describes the elementary displacements dX^k, $k = 1, 2, 3$, on the geodesics of a nondifferentiable fractal space-time in terms of the sum of two terms (omitting the indices for simplicity) $dX_\pm = d_\pm x + d\xi_\pm$, where dx represents the "classical (differentiable) part" and $d\xi$ the "fractal (nondifferentiable) part", defined as

$$d_\pm x = v_\pm \; dt, \quad d\xi_\pm = \eta \sqrt{2\mathcal{D}}\, dt^{1/2}, \tag{7}$$

where η is a stochastic dimensionless variable such that $< \eta >= 0$ and $< \eta^2 >= 1$, and $\mathcal{D}$ is a parameter that generalizes, up to the fundamental constant $c/2$, the Compton scale ($\mathcal{D} = \hbar/2m$ in the case of standard quantum mechanics). The two time derivatives are then combined in terms of a complex derivative [1],

$$\frac{\widehat{d}}{dt} = \frac{1}{2}\left(\frac{d_+}{dt} + \frac{d_-}{dt}\right) - \frac{i}{2}\left(\frac{d_+}{dt} - \frac{d_-}{dt}\right) . \tag{8}$$

Applying this operator to the position vector yields a complex velocity

$$\mathcal{V} = \frac{\widehat{d}}{dt} x(t) = V - iU = \frac{v_+ + v_-}{2} - i\,\frac{v_+ - v_-}{2} . \tag{9}$$

Then one writes a geodesics equation, $\widehat{d}\mathcal{V}/dt = 0$, which can be integrated under the form of a generalized Schrödinger equation [1, 2, 9, 13, 14]. However, in such a derivation, only the classical part of the velocity is taken into account when defining the wave function. We have recently generalized the proof to the whole velocity field, including its divergent (nondifferentiable) part [20, 17].

Let us indeed now consider the full complex velocity field, including its differentiable and nondifferentiable parts,

$$\widetilde{\mathcal{V}} = \mathcal{V} + \mathcal{W} = \left(\frac{v_+ + v_-}{2} - i\,\frac{v_+ - v_-}{2}\right) + \left(\frac{w_+ + w_-}{2} - i\,\frac{w_+ - w_-}{2}\right). \tag{10}$$

From it we can build a full complex action,

$$d\tilde{\mathcal{S}} = \frac{1}{2} m (\mathcal{V} + \mathcal{W})^2 \, dt, \tag{11}$$

then one defines a full wavefunction $\tilde{\psi}$ from this full action $\tilde{\mathcal{S}}$ as

$$\tilde{\psi} = e^{i\tilde{\mathcal{S}}/2m\mathcal{D}}. \tag{12}$$

It is linked to the full complex velocity by the relation

$$\widetilde{\mathcal{V}} = \mathcal{V} + \mathcal{W} = \nabla\tilde{\mathcal{S}}/m = -2i\mathcal{D}\nabla \ln \tilde{\psi}. \tag{13}$$

Under the standard point of view, the complex fluctuation $\mathcal{W}$ is infinite and therefore $\nabla \ln \tilde{\psi}$ is undefined, so that equation (13) would be meaningless. In the scale relativity approach, on the contrary, this equation keeps a mathematical and physical meaning, in terms of fractal functions, which are explicitly dependent on the scale interval dt and divergent when $dt \to 0$.

The fractal parts of the velocities may be written under the form:

$$w_+ = \eta_+ \sqrt{\frac{2\mathcal{D}}{dt}}, \quad w_- = \eta_- \sqrt{\frac{2\mathcal{D}}{dt}}, \tag{14}$$

where η_+ and η_- are stochastic variables such that $< \eta_+ >=< \eta_- >= 0$ and $< \eta_+^2 >=< \eta_-^2 >= 1$. The(+) and (−) derivatives read

$$\frac{d_\pm f}{dt} = \frac{\partial f}{\partial t} + (v_\pm + w_\pm)\nabla f + \mathcal{D}\,\eta_\pm^2\,\Delta f + ..., \tag{15}$$

where the next terms are infinitesimals. Let us define the following complex stochastic variables:

$$\tilde{\eta} = \frac{\eta_+ + \eta_-}{2} - i\,\frac{\eta_+ - \eta_-}{2}, \quad 1 + \tilde{\zeta} = \frac{\eta_+^2 + \eta_-^2}{2} + i\,\frac{\eta_+^2 - \eta_-^2}{2}, \tag{16}$$

which are such that $< \tilde{\eta} >= 0$ and $< \tilde{\zeta} >= 0$. We can now combine the two derivatives in terms of a generalized complex covariant derivative,

$$\frac{\widehat{d}}{dt} = \frac{\partial}{\partial t} + (\mathcal{V} + \mathcal{W}).\nabla - i\mathcal{D}(1 + \tilde{\zeta})\Delta, \tag{17}$$

plus infinitesimal terms that vanish when $dt \to 0$. We therefore recover the mean covariant derivative introduced in previous works [1, 2, 9], namely $\widehat{d}/dt = \partial/\partial t + \mathcal{V}.\nabla - i\mathcal{D}\Delta$, plus two additional stochastic terms of zero mean. The first of these terms is $\mathcal{W}.\nabla$, which is infinite at the limit $dt \to 0$. The second is $-i\mathcal{D}\,\tilde{\zeta}\,\Delta$, in which $\tilde{\zeta}$ remains finite, so that it can be neglected as was done in Ref. [20], since their ratio is an infinitesimal of order $dt^{1/2}$.

Using this covariant derivative, we can finally write a full equation of motion for a free "particle" in terms of a geodesics equation, namely, that keeps the form of the free Galilean inertial motion equation [20], $\widehat{d}\,\widetilde{\mathcal{V}}/dt = 0$. In the presence of a potential ϕ, it can be easily generalized in terms of a covariant equation which keeps the form of Newton's fundamental equation of dynamics,

$$\frac{\widehat{d}}{dt}\widetilde{\mathcal{V}} = -\frac{\nabla\phi}{m}. \tag{18}$$

After expansion of this equation and replacement of the velocity field by its expression in terms of the wave function, we obtain [20, 17]

$$\left(\frac{\partial}{\partial t} + (-2i\mathcal{D}\nabla \ln \tilde{\psi}).\nabla - i\mathcal{D}\Delta\right)(-2i\mathcal{D}\nabla \ln \tilde{\psi}) = -\frac{\nabla\phi}{m}. \tag{19}$$

We are in the same conditions (but now using fractal functions) as in previous calculations involving a differentiable wave function [1, 9], so that this equation can finally be integrated in terms of a generalized Schrödinger equation,

$$\mathcal{D}^2\Delta\tilde{\psi} + i\mathcal{D}\,\frac{\partial\tilde{\psi}}{\partial t} - \frac{\phi}{2m}\tilde{\psi} = 0. \tag{20}$$

This generalized Schrödinger equation now allows fractal solutions, which come, in our framework, as a direct manifestation of the nondifferentiability of space. Such a result agrees with Berry's [18] and Hall's [19] findings obtained in the framework of standard quantum mechanics. The research of such a behavior in laboratory experiments is an interesting new challenge for quantum physics.

2.4 Derivation of Von Neumann's and Born's postulates

We have identified the "particle" with the various geometric properties of fractal space-time geodesics. In such an interpretation, a measurement (and more generally any knowledge about the system) amounts to a selection of the sub-sample of the geodesics family in which are kept only the geodesics having the geometric properties corresponding to the measurement result. Therefore, just after the measurement, the system is in the state given by the measurement result, which is precisely the von Neumann postulate of quantum mechanics.

The Born postulate can now also be inferred from the scale-relativity construction [9, 17]. Indeed, the probability for the "particle" to be found at a given position must be proportional to the density of the geodesics fluid. The velocity and the density of this fluid are expected to be solutions of a Euler + continuity system of four equations, for four unknowns, (ρ, V_x, V_y, V_z).

Now, by separating the real and imaginary parts of the Schrödinger equation, setting $\psi = \sqrt{P} \times e^{i\theta}$ and using a mixed representation (P, V), one obtains precisely such a standard system of fluid dynamics equations, namely,

$$\left(\frac{\partial}{\partial t} + V \cdot \nabla\right) V = -\nabla \left(\phi - 2\mathcal{D}^2 \frac{\Delta\sqrt{P}}{\sqrt{P}}\right), \qquad \frac{\partial P}{\partial t} + \mathrm{div}(PV) = 0. \tag{21}$$

This allows one to univoquely identify $P = |\psi|^2$ with the probability density of the geodesics and therefore with the probability of presence of the 'particle'. Moreover,

$$Q = -2\mathcal{D}^2 \frac{\Delta\sqrt{P}}{\sqrt{P}} \tag{22}$$

can be interpreted as the new potential which is expected to emerge from the fractal geometry [15], in analogy with the identification of the gravitational field as a manifestation of the curved geometry in general relativity. This result is supported by numerical simulations, in which the probability density is obtained directly from the distribution of geodesics without writing the Schrödinger equation [16].

2.5 Dirac and Pauli equations

All these results can be generalized to relativistic quantum mechanics, that corresponds in the scale relativity framework to a full fractal space-time. This yields, as a first step, the Klein-Gordon equation [12, 2].

Then the account of a new two-valuedness of the velocity allows one to suggest a geometric origin for the spin and to obtain the Dirac equation [9]. Indeed, the total derivative of a physical quantity also involves partial derivatives with respect to the space variables, $\partial/\partial x^\mu$. From the very definition of derivatives, the discrete symmetry under the reflection $dx^\mu \leftrightarrow -dx^\mu$ is also broken. Since, at this level of description, one should also account for parity, as in the standard quantum theory, we have been led to introduce a bi-quaternionic velocity field [9].

The successive steps that lead to the Dirac equation naturally generalize the Schrödinger case. One introduces a biquaternionic generalization of the covariant derivative that keeps the same form as in the complex case, namely,

$$\frac{\widehat{d}}{ds} = \mathcal{V}^\nu \partial_\nu + i\,\frac{\lambda}{2}\,\partial^\nu \partial_\nu, \tag{23}$$

where $\lambda = 2\mathcal{D}/c$. The biquaternionic velocity field is related to the biquaternionic, i.e. bispinorial, wave function, as

$$\mathcal{V}_\mu = i\frac{S_0}{m}\,\psi^{-1}\partial_\mu \psi. \tag{24}$$

The covariance principle allows us to write the equation of motion under the form of a geodesics differential equation,

$$\frac{\widehat{d}\,\mathcal{V}_\mu}{ds} = 0. \tag{25}$$

After some calculations, this equation can be integrated and factorized, and one finally derives the Dirac equation [9],

$$\frac{1}{c}\frac{\partial \psi}{\partial t} = -\alpha^k \frac{\partial \psi}{\partial x^k} - i\frac{mc}{\hbar}\beta\psi. \tag{26}$$

Finally it is easy to recover the Pauli equation and Pauli spinors as nonrelativistic approximation of the Dirac equation and Dirac bispinors [21, 22].

3 Gauge theories in scale relativity

3.1 General scale transformations and gauge fields

Let us now briefly recall the main steps of the application of scale relativity to the foundation of gauge theories, in the Abelian [12, 2] and non-Abelian [23] cases.

This application is based on a general description of the internal fractal structures of the "particle" (identified with nondifferentiable space-time geodesics) in terms of scale variables $\eta_{\alpha\beta}(x, y, z, t)$ whose true nature is tensorial and may now be function of the coordinates. This resolution tensor (similar to a covariance error matrix) generalizes the single resolution variable ε. This means that we are now in the framework of a general scale relativity.

We assume for simplicity of the writing that the two tensorial indices can be gathered under one common index. We therefore write the scale variables under the simplified form $\eta_{\alpha_1\alpha_2} = \eta_\alpha$, $\alpha = 0$ to $n(n+1)/2$, where n is the number of space-time dimensions ($n = 3$ for fractal space, 4 for fractal space-time and 5 in the special scale relativity case [7]).

Let us consider infinitesimal scale transformations. The transformation law on the η_α can be written in a linear way as

$$\eta'_\alpha = \eta_\alpha + \delta\eta_\alpha = (\delta_{\alpha\beta} + \delta\theta_{\alpha\beta})\,\eta^\beta, \tag{27}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol. Let us now assume that the η_α's are functions of the standard space-time coordinates. This leads us to define a new scale-covariant derivative by writing the total variation of the resolution variables as the sum of the inertial variation, described by the covariant derivative, and of the new geometric contribution, namely,

$$d\eta_\alpha = D\eta_\alpha - \eta^\beta \delta\theta_{\alpha\beta} = D\eta_\alpha - \eta^\beta W^\mu_{\alpha\beta}\, dx_\mu. \tag{28}$$

This covariant derivative is similar to that of general relativity, i.e., it amounts to substract the new geometric part in order to keep only the inertial part (for which the motion equation will therefore take a geodesical, freelike form). This is different from the case of the previous quantum-covariant derivative, which includes the effects of nondifferentiability by adding new terms in the total derivative.

In this new situation we are led to introduce "gauge field potentials" $W^{\mu}_{\alpha\beta}$ that enter naturally in the geometrical framework of Eq. (28). These potentials are linked to the scale transformations as follows:

$$\delta\theta_{\alpha\beta} = W^{\mu}_{\alpha\beta}\, dx_{\mu}. \tag{29}$$

One should keep in mind, when using this expression, that these potentials find their origin in a covariant derivative process and are therefore not gradients.

3.2 General charges

After having written the transformation law of the basic variables (the η_α's), we now need to describe how various physical quantities transform under these η_α transformations. These new transformation laws are expected to depend on the nature of the objects to transform (e.g., vectors, tensors, spinors, etc.), which implies to jump to group representations.

We anticipate the existence of charges (which are fully constructed herebelow) by generalizing to multiplets the relation (Eq. 24) between the velocity field and the wave function. In this case the multivalued velocity becomes a biquaternionic matrix,

$$\mathcal{V}^{\mu}_{jk} = i\lambda\, \psi_j^{-1}\partial^{\mu}\psi_k. \tag{30}$$

The biquaternionic (therefore noncommutative) nature of the wave function, which is equivalent to Dirac bispinors, plays here an essential role. Indeed, the general structure of Yang-Mills theories and the correct construction of non-Abelian charges can be obtained thanks to this result [23].

The action also becomes a tensorial biquaternionic quantity,

$$dS_{jk} = dS_{jk}(x^{\mu}, \mathcal{V}^{\mu}_{jk}, \eta_\alpha), \tag{31}$$

and, in the absence of a field, it is linked to the generalized velocity (and therefore to the spinor multiplet) by the relation

$$\partial^{\mu} S_{jk} = -mc\, \mathcal{V}^{\mu}_{jk} = -i\hbar\, \psi_j^{-1}\partial^{\mu}\psi_k. \tag{32}$$

Now, in the presence of a field (i.e., when the second-order effects of the fractal geometry appearing in the right hand side of Eq. (28) are included), using the complete expression for $\partial^{\mu}\eta_\alpha$,

$$\partial^{\mu}\eta_\alpha = D^{\mu}\eta_\alpha - W^{\mu}_{\alpha\beta}\, \eta^{\beta}, \tag{33}$$

we obtain a non-Abelian relation,

$$\partial^{\mu} S_{jk} = D^{\mu} S_{jk} - \eta^{\beta}\, \frac{\partial S_{jk}}{\partial \eta_\alpha}\, W^{\mu}_{\alpha\beta}. \tag{34}$$

We are finally led to define a general group of scale transformations whose generators are

$$T^{\alpha\beta} = \eta^{\beta}\partial^{\alpha} \tag{35}$$

(where we use the compact notation $\partial^{\alpha} = \partial/\partial\eta_\alpha$), yielding the generalized charges,

$$\frac{\tilde{g}}{c}\, t^{\alpha\beta}_{jk} = \eta^{\beta}\, \frac{\partial S_{jk}}{\partial \eta_\alpha}. \tag{36}$$

This unified group is submitted to a unitarity condition, since, when it is applied to the wave functions, $\psi\psi^\dagger$ must be conserved. Knowing that the α, β represent two indices each, this is a large group that contains the standard model $U(1) \times SU(2) \times SU(3)$ as a subset [23].

As we have shown in more detail in Ref. [23], the various ingredients of Yang-Mills theories (gauge covariant derivative, gauge invariance, charges, potentials, fields, etc...) may subsequently be recovered in such a framework, but they now have a first principle and geometric scale-relativistic foundation.

4 Conclusion

In this contribution, we have recalled the main steps that lead to a new foundation of quantum mechanics and of gauge fields on the principle of relativity itself, once it is generalized to scale transformations of the reference system.

For this purpose, two covariant derivatives have been constructed, which account for the nondifferentiable and fractal geometry of space-time, and which allow to write the equations of motion as geodesics equations. After change of variable, these equations finally take the form of the quantum mechanical and quantum field equations.

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