Geometric interpretation of gauge fields in the theory of scale relativity*

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Abstract

The question of the physical nature of gauge fields is revisited in the framework of the theory of scale relativity, first in the Abelian case (quantum electrodynamics), then in the non-Abelian case (SU(2) and more general fields). Space-time is described as a non-differentiable and fractal continuous manifold (i.e., it is explicitly dependent on the scale of resolution). The gauge fields can then be interpreted in a purely geometric way: they are the manifestations of the contraction and dilation of the scale variables induced by space-time displacements, at scales smaller than the Compton length of a particle. The theory also allows one to give a geometrical meaning to the charges: they are defined as the conservative quantities that originate from the new scale symmetries. Moreover, in the framework of special scale-relativity, where the Planck length becomes a minimal, impassable horizon-like scale, invariant under dilations, the quantization of charges is ensured because the possible scale ratios become limited. As a consequence we theoretically predict relations between coupling constants α and mass scales m of the form $\alpha \ln(m_{\mathbb{P}}/m) = 1$, where $m_{\mathbb{P}}$ is the Planck mass. Some evidence for the existence of such relations in the experimental data is given. We conclude by recalling our theoretical prediction of a Higgs boson mass of $\sqrt{2}\, m_W = 114$ GeV.

1 Introduction

In its present state, fundamental physics can be considered as still unsatisfactory. In particular, despite impressive progress, the theory of gauge fields (at first, electromagnetism) cannot be considered to be as well founded as the theory of gravitation is in the framework of Einstein's general relativity.

Indeed, in the present physical theory, one still does not really understand form first principles neither the nature of the electric charge and of the electromagnetic field, nor the

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fundamental meaning of gauge invariance. As recalled by Landau [1], in the classical theory the very existence of the charge e and of the electromagnetic 4-potential A_μ are ultimately derived from experimental data. Moreover, the form of the action that describes the coupling of a particle with an electromagnetic field can not be chosen only from general considerations, and it is therefore merely postulated.

Now, once these three points are set (the existence of the charge, that of the electromagnetic field and the minimal coupling between them), most of the classical theory of electromagnetism can be constructed. Indeed (see e.g. [1]), the Lorentz force and the Maxwell equations can subsequently be derived from the action principle. But the arbitrary character of the coupling term in the action is another expression for the fact that the motion equation cannot be derived from the field equations, so that they have been added by Lorentz in an independent way.

This is to be compared with the status of gravitation in Einstein's theory [2]. The 'charge' for gravitation is the energy-momentum itself, which is fully understood as the conservative quantities that take their origin in the fundamental symmetries of space-time (following Noether's theorem). Then the gravitational field is understood as the manifestation of a geometric property of space-time, namely, its curvature, which is itself self-imposed from the principle of general relativity. Finally, there is no need to add an independent equation of trajectories to the theory, since it is identified with the equation of geodesics, which is completely determined by the knowledge of the space-time geometry.

Therefore the present state of the foundation of the classical electromagnetic theory remains far less satisfactory than the theory of gravitation in Einstein's general relativity. This remains a standard field theory, not a geometric theory based on the principle of relativity. Up to now, the various attempts of foundation of electromagnetism on a space-time approach (Kaluza-Klein [5], Weyl [3], Dirac [4], etc...) have failed to yield new results that would have allowed to validate or refute them.

More recently, the quantum theory of electromagnetism and of the electron has added a new and essential stone in our understanding of the nature of charge. Indeed, in its framework, gauge invariance becomes deeply related to phase invariance of the wave function. The electric charge conservation is therefore directly related to the gauge symmetry. Such an understanding has led to important progress: in particular, the extension of the approach to non-Abelian gauge theories has allowed to incorporate the weak and strong field into the same scheme.

However, despite this progress, the lack of a fundamental understanding of the nature of the gauge transformations and of the 'arbitrary' gauge functions has up to now prevented from reaching the final goal of gauge theories: namely, to reach a genuine understanding of the nature of charge, to understand why charge is quantized and, as a consequence, to be able to theoretically calculate its quantized value.

The theory of scale relativity [6, 7, 8], in which we generalize the geometry of space-time to continous but non-differentiable geometries, allows one to reconsider this problem in its framework.

In the present contribution, we first summarize the principles that underly the theory of scale relativity (Sec. 3). Then we recall how one can recover standard self-similar scale laws with constant fractal dimension as the simplest expression of the principle of scale-relativity in the 'Galilean ' case. However, even in that case we find a spontaneous symmetry breaking of the scale dependence toward the large scales at a transition scale that is identified with the Compton-Einstein-de Broglie scale. In other words, this means that space-time appears not to be fractal at all scales, but only in the quantum regime, while we recover the classical space-time toward the large scales. The generalisation of these scale laws to a Lorentz group of scale transformations (in which the Planck length becomes invariant under dilations) is subsequently described. It is in the framework of this special scale-relativity theory that the electron charge becomes quantized and that its value becomes related to its mass. Then we briefly review the scale-relativistic approach to the quantum theory, recalling how one can derive the Klein-Gordon and Dirac equations from its principles. We also show how the scale-relativity tools can be improved in order to fully implement the strong covariance / equivalence principle (according to which the equations of physics can be written under the form of free, inertial-like equations).

In Sec. 4 we develop the classical theory of electrodynamics in the scale-relativity framework. Classical gauge invariance is recovered. Geometric definitions for the electromagnetic field and for the electric charge are given. Then the equation of motion of a charged particle in an electromagnetic field is constructed in terms of a geodesics equation in a fractal space-time.

In Sec. 5 we combine the various tools of the scale-relativity approach, that have given rise to the quantum description on one hand and to classical electrodynamics on the other. The covariant derivative of QED is constructed in a geometric way. Then we write a doubly covariant free-like geodesics equation, which can be finally integrated in terms of the Klein-Gordon and Dirac equations for a particle in an electromagnetic field. As a consequence of this approach, the electric charge becomes quantized in the special-scale-relativity case, and a relation between the mass and the charge of the electron is theoretically expected.

In Sec. 6 some of these results are generalized in a non-Abelian framework. A general geometric definition of the gauge fields and of their associated charges can be given, which allows one to recover the standard tools of Yang-Mills theories. We conclude by preliminary applications of this approach to mass-coupling relations and to a theoretical prediction of the Higgs boson mass.

2 Statement of the problem

Consider an electron in an electromagnetic potential A_μ. This potential is invariant under a gauge tansformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi(x, y, z, t)$. Let us consider the wave function of an electron. It writes:

$$\psi = \psi_0 \ \exp\left\{ \frac{i}{\hbar}(px - Et + \sigma\varphi + e\chi) \right\} \tag{1}$$

Its phase contains the usual products of fundamental quantities (space position, time, angle) and of their conjugate quantities (momentum, energy, angular momentum). They are related through Noether's theorem. Namely, the conjugate variables are the conservative quantities that originate from the space-time symmetries. This means that our knowledge of what are the energy, the momentum and the angular momentum and of their physical properties is founded on our knowledge of the nature of space, time and its transformations (translations and rotations).

This is true already in the classical theory, but there is something more in the quantum theory. In its framework, the conservative quantities are quantized when the basic variables are limited. Concerning energy-momentum, this means that it is quantized only in some specific circumstances (e.g., bound states in atoms for which $r > 0$ in spherical coordinates). The case of the angular momentum is instructive: its differences are quantized in an universal way in units of $\hbar$ because angles differences can not exceed 2π.

In comparison, the last term in the phase of Eq. (1) keeps a special status in today's standard theory. The gauge function $\chi(x, y, z, t)$ remains arbitrary, while it is clear from a comparison with the other terms that the meaning of charge e and the reason for its universal quantization can be obtained only by understanding the physical meaning of χ and why it is universally limited, since it is nothing but the quantity conjugate to the charge. As we shall see in what follows, the identification of χ with a resolution scale factor $\ln \varrho$ allows one to suggest solutions to these problems in the special scale-relativity framework [9].

3 The theory of scale relativity: summary

3.1 Principle of scale relativity

The theory of scale relativity [8] studies the consequences of giving up the hypothesis of space-time differentiability. One can show [8, 10, 11] that a continuous but nondifferentiable space-time is necessarily fractal. Here the word fractal [12, 13] is taken in a general meaning, as defining a set, object or space that shows structures at all scales, or on a wide range of scales. More precisely, one can prove [10] that a continuous but nondifferentiable function is explicitly resolution-dependent, and that its length $\mathcal{L}$ is strictly increasing and tends to infinity when the resolution interval tends to zero, i.e. $\mathcal{L} = \mathcal{L}(\varepsilon)_{\varepsilon \to 0} \to \infty$. This theorem naturally leads to the proposal that the concept of fractal space-time [14, 15, 6, 8, 16] is the geometric tool adapted to the research of such a new description.

Since a nondifferentiable, fractal space-time is explicitly resolution-dependent, the same is a priori true of all physical quantities that one can define in its framework. We thus need to complete the standard laws of physics (which are essentially laws of motion in classical physics) by laws of scale, intended to describe the new resolution dependence. We have suggested [7] that the principle of relativity can be extended to constrain also these new scale laws.

Namely, we generalize Einstein's formulation of the principle of relativity, by requiring that the laws of nature be valid in any reference system, whatever its state. Up to now, this principle has been applied to changes of state of the coordinate system that concerned the origin, the axes orientation, and the motion (measured in terms of velocity and acceleration).

In scale relativity, we assume that the space-time resolutions are not only a characteristic of the measurement apparatus, but acquire a universal status. They are considered as essential variables, inherent to the physical description. We define them as characterizing the "state of scale" of the reference system, in the same way as the velocity characterizes its state of motion. The principle of scale relativity consists of applying the principle of relativity to such a scale-state. Then we set a principle of scale-covariance, requiring that the equations of physics keep their form under resolution transformations.

3.2 Galilean scale-relativity

Simple fractal scale-invariant laws can be identified with a "Galilean" version of scale-relativistic laws. Indeed, let us consider a non-differentiable coordinate $\mathcal{L}$. Our basic theorem that links non-differentiability to fractality implies that $\mathcal{L}$ is an explicit function $\mathcal{L}(\varepsilon)$ of the resolution interval ε. As a first step, one can assume that $\mathcal{L}(\varepsilon)$ satisfies the simplest possible scale differential equation one may write, namely, a first order equation where the scale variation of $\mathcal{L}$ depends on $\mathcal{L}$ only, $d\mathcal{L}/d\ln\varepsilon = \beta(\mathcal{L})$. The function $\beta(\mathcal{L})$ is a priori unknown but, still taking the simplest case, we may consider a perturbative approach and take its Taylor expansion. We obtain the equation:

$$\frac{d\mathcal{L}}{d\ln\varepsilon} = a + b\mathcal{L} + \ldots \tag{2}$$

This equation is solved in terms of a standard power law of power $\delta = -b$, broken at some relative scale λ (which is a constant of integration):

$$\mathcal{L} = \mathcal{L}_0 \left[1 + \left(\frac{\lambda}{\varepsilon}\right)^{\delta}\right]. \tag{3}$$

Here δ is the scale dimension, i.e., $\delta = D_F - D_T$, the fractal dimension minus the topological dimension. The scale symmetry breaking at the transition scale λ plays an important role in the theory, since this scale is subsequently identified with the Einstein-de Broglie scale, so that it ultimately gives a new interpretation of the mass of the particle under consideration as a purely geometric quantity.

One can easily verify that, under a scale transformation $\varepsilon \to \varepsilon'$, the variables $\ln\mathcal{L}$ and δ transform in the asymptotic domain according to a Galilean scale group, which establishes that they come under the principle of scale relativity [7, 8].

3.3 Special scale-relativity

We have suggested that, in the fractal asymptotic domain, (i.e. beyond the fractal / non-fractal transition λ, that is identified in rest frame with the Compton length of the particle), the Galilean law of composition of dilations $\ln(\varepsilon'/\lambda) = \ln\rho + \ln(\varepsilon/\lambda)$ is only a low energy approximation, and should be replaced by the more general log-Lorentzian law [7]:

$$\ln\frac{\varepsilon'}{\lambda} = \frac{\ln\rho + \ln(\varepsilon/\lambda)}{1 + \ln\rho\,\ln(\varepsilon/\lambda)/\ln^2(\lambda_{I\!P}/\lambda)}\,. \tag{4}$$

In the framework of such a 'special scale-relativistic' law, the length-time scale $\lambda_{\mathbb{P}}$ is a minimal scale of space-time resolution which is invariant under dilations and contractions, and plays the same role for scales as that played by the velocity of light for motion.

Toward the small scales, this invariant length-scale is naturally identified with the Planck scale, $\lambda_{\mathbb{P}} = (\hbar G/c^3)^{1/2}$, that now becomes impassable and plays the physical role that was previously devoted to the zero point. Some consequences of this new interpretation of the Planck length-time-scale have been considered elsewhere [7, 8, 10], concerning in particular the unification of fundamental fields. We shall point out here its consequences for the quantization of the electric charge and for a theoretical prediction of a relation between the electric charge and the electron mass.

3.4 Relativistic quantum mechanics: Klein-Gordon equation

One of the main consequences of the scale-relativity / fractal space-time approach is its ability to build from geometric structures the fundamental rules of quantum mechanics [8, 10, 17], that were up to now set as mere axioms.

Let us first briefly recall here how one can establish the free Klein-Gordon equation in the scale-relativity framework [9, 10].

The space-time continuum is described in terms of a nondifferentiable continuum, which is therefore fractal [8]. Due to the fractality of space-time, the number of geodesics is infinite. This leads one to introduce a velocity field instead of a deterministic velocity and to work in terms of a fluid-like approach.

Due to the breaking of the reflection symmetry ($ds \leftrightarrow -ds$) issued from non-differentiability, the elementary displacements dX^μ become two-valued [8, 17], so that we define dX^μ_+ and dX^μ_-.

According to Eq. 3, the elementary displacements are decomposed in terms of a differentiable 'classical' part and of a non-differentiable fractal part (here in the standard case of a fractal dimension $D_F = 2$ of trajectories):

$$dX^\mu_\pm = dx^\mu_\pm + d\xi^\mu_\pm. \tag{5}$$

We are lead to a stochastic description, due to the infinity of geodesics of the fractal space-time. The non-differentiable part of the infinitesimal displacement is described in terms of two fluctuation fields, $d\xi^\mu_\pm(s)$, which have zero expectation, $< d\xi^\mu_\pm >= 0$, are mutually independent and such that

$$< d\xi^\mu_\pm \, d\xi^\nu_\pm >= \mp\lambda \, \delta^{\mu\nu} ds. \tag{6}$$

The constant λ identifies with the Compton length of the particle: in other words, this yields a new definition of the rest mass, which is here generated by the infinite fractal fluctuations. The "metric" $\delta^{\mu\nu}$ is positive definite: indeed, the fractal fluctuations are of the same nature as uncertainties and 'errors', so that the space and the time fluctuations add quadratically.

One defines a two-valued 'classical' derivative, d_+/ds and d_-/ds. Once applied to x^μ, it yields two classical 4-velocities,

$$\frac{d_+}{ds} x^\mu(s) = v_+^\mu \quad ; \quad \frac{d_-}{ds} x^\mu(s) = v_-^\mu. \tag{7}$$

The two derivatives of Eq.(7) can be combined in terms of a complex derivative operator

$$\frac{\acute{d}}{ds} = \frac{(d_+ + d_-) - i\,(d_+ - d_-)}{2ds}, \tag{8}$$

which, when applied to the position vector, yields a complex 4-velocity

$$\mathcal{V}^\mu = \frac{\acute{d}}{ds} x^\mu = V^\mu - i\,U^\mu = \frac{v_+^\mu + v_-^\mu}{2} - i\,\frac{v_+^\mu - v_-^\mu}{2}. \tag{9}$$

The two (+) and (−) differentials of a function $f[x(s)]$ can now be written:

$$d_\pm f/ds = (v_\pm^\mu\,\partial_\mu \mp \frac{1}{2}\lambda \partial^\mu \partial_\mu) f. \tag{10}$$

so that the 'quantum-covariant' derivative operator finally reads:

$$\frac{\acute{d}}{ds} = \left(\mathcal{V}^\mu + \frac{1}{2} i\,\lambda\,\partial^\mu\right)\partial_\mu. \tag{11}$$

We use here the expression 'covariant' as an extension of Einstein's definition, i.e. as a tool that allows one to keep the form of equations in a new and more complicated situation. But one should remark that the true status of this covariant derivative is actually an extension of the concept of total derivative. Already in standard physics, the passage from the free Galileo-Newton's equation to its Euler form was a case of conservation of the form of equations in a more complicated situation, namely, $d/dt \to \partial/\partial t + v.\nabla$. In the fractal and non-differentiable situation considered here, two new terms appear in the total derivative operator, namely $-i\,U^\mu \partial_\mu$ (where $-U^\mu$ is the imaginary part of $\mathcal{V}^\mu$) and $-i(\lambda/2)\partial^\mu\partial_\mu$. These terms are added because the differential calculus itself is affected, and they yield the quantum behavior. As we shall recall in what follows, we are led to introduce another covariant derivative more similar to that of general relativity, that describes the effects of the fractal geometry, which manifests itself in terms of the gauge fields [9, 10, 18].

Since the velocity is now complex, the same is true of the action $\mathcal{S}$. The wave function is introduced as being nothing but its reexpression:

$$\psi = e^{i\mathcal{S}/mc\lambda} \;\Rightarrow\; \mathcal{V}_\mu = i\lambda\partial_\mu(\ln\psi), \tag{12}$$

Using the quantum-covariant derivative, we finally write the equations of motion under the form of a covariant, free-like, geodesics equation:

$$\acute{d}\,\mathcal{V}_\mu/ds = 0 \tag{13}$$

By replacing the covariant derivative operator by its expression we obtain:

$$\left(\mathcal{V}^\mu + \frac{1}{2} i\lambda\partial^\mu\right)\partial_\mu \mathcal{V}_\alpha = 0. \tag{14}$$

Arrived at this stage, it is easy to show that this equation amounts to the Klein-Gordon one. Introducing the wave function and accounting for the identity [8]

$$\partial_\mu\partial^\mu \ln\psi + \partial_\mu \ln\psi \, \partial^\mu \ln\psi = \frac{\partial_\mu\partial^\mu\psi}{\psi}, \tag{15}$$

it can be integrated in terms of the Klein-Gordon equation for a free particle:

$$\partial_\nu\left(\frac{\lambda^2\partial^\mu\partial_\mu\psi}{\psi}\right) = 0 \;\Rightarrow\; \lambda^2\partial^\mu\partial_\mu\psi = \psi. \tag{16}$$

3.5 Dirac equation

More recently, it has been shown [17] that the Dirac equation could also be derived from the same principles. We have seen that, in the scale-relativity approach, the complex nature of the wave function in quantum mechanics is a direct consequence of the non-differentiable geometry of space-time, which involves a symmetry breaking of the reflection invariance $dt \leftrightarrow -dt$, and therefore a two-valuedness of the classical velocity vector.

Going to motion-relativistic quantum mechanics amounts to introduce not only a fractal space, but a fractal space-time. The invariant parameter becomes in this case the proper time s instead of the time t. As a consequence the complex nature of the four-dimensional wave function in the Klein-Gordon equation comes from the discrete symmetry breaking $ds \leftrightarrow -ds$.

Now, the total derivative of a physical quantity also involves partial derivatives with respect to the space variables, $\partial/\partial x^\mu$. Once again, from the very definition of derivatives, the discrete symmetry under the reflection $dx^\mu \leftrightarrow -dx^\mu$ should also be broken at a more profound level of description. Therefore, we expect the appearance of a new two-valuedness of the generalized velocity.

At this level one should also account for parity violation. Finally, we have suggested that the three discrete symmetry breakings

$$ds \leftrightarrow -ds \qquad dx^\mu \leftrightarrow -dx^\mu \qquad x^\mu \leftrightarrow -x^\mu$$

can be accounted for by the introduction of a bi-quaternionic velocity, which is at the origin of the bi-spinor nature of the electron wave function. It has been subsequently shown [17] that one can derive in this way the Dirac equation, namely as an integral of a geodesics equation.

One can indeed show that, despite their noncommutativity, one recovers Eq. (16) in terms of biquaternions. Then, when one expands the biquaternionic form of this Klein-Gordon equation, one finds that it is spontaneously written as the square of the Dirac equation [19, 17],

$$\frac{1}{c}\frac{\partial\psi}{\partial t}+\alpha^k\frac{\partial\psi}{\partial x^k}-i\frac{mc}{\hbar}\beta\psi=0, \tag{17}$$

which is therefore demonstrated.

3.6 Covariant tool: complex velocity operator

The next step amounts to mix the two consequences of the nondifferentiable / fractal geometry in order to reach a construction of quantum electrodynamics from first principles. In this purpose, we need to first develop further the quantum-covariant tool of scale-relativity.

As shown by Zastawniak [20] and as can be easily recovered from the definition (9), the quadratic invariant of special motion-relativity, $v^\mu v_\mu = 1$, is naturally generalized as $\mathcal{V}^\dagger_\mu \mathcal{V}^\mu = 1$ where $\mathcal{V}^\dagger_\mu$ is the complex conjugate of $\mathcal{V}_\mu$. Concerning the square of the four-velocity, it is now complex. It has been shown by Pissondes [21] that it now reads

$$\mathcal{V}_\mu\mathcal{V}^\mu + i\lambda\partial_\mu\mathcal{V}^\mu = 1, \tag{18}$$

which is a direct consequence of the identity (15). Now taking the gradient of this equation, one recovers another form of Eq. (14):

$$\left(\mathcal{V}^\mu+\frac{1}{2}i\lambda\partial^\mu\right)\partial_\alpha\mathcal{V}_\mu=0. \tag{19}$$

In the free case the two equations are equivalent, since $\partial_\alpha\mathcal{V}_\mu-\partial_\mu\mathcal{V}_\alpha=0$.

Now, the basic definition of the quantum-covariant derivative of a function $f[x(s)]$ is a linear combination of a first order and of a second order derivative (a Laplacian):

$$\frac{\acute{d}}{ds}f[x(s)]=\mathcal{V}^\mu\partial_\mu f+i\frac{\lambda}{2}\partial^\mu\partial_\mu f. \tag{20}$$

Although $\mathcal{V}^\mu$ is now complex, the first term keeps the form of the standard derivative of a composed function, $df/ds = u^\mu\partial_\mu f$. But the second term implies to be cautious when using this operator in calculations. In particular, the derivative of a product will be based on a combination of the Leibniz rules for first order derivatives and also for second order derivatives, which is non-linear. Indeed one finds [21]:

$$\frac{\acute{d}}{ds}(fg)=f\frac{\acute{d}g}{ds}+g\frac{\acute{d}f}{ds}+i\lambda\,\partial^\mu f\,\partial_\mu g. \tag{21}$$

In order to keep the form of the first order Leibniz rule (i.e. to implement strong covariance), Pissondes [21] has defined a 'symmetric' product

$$f\circ\frac{\acute{d}g}{ds}=f\frac{\acute{d}g}{ds}+i\frac{\lambda}{2}\,\partial^\mu f\,\partial_\mu g, \tag{22}$$

and he has subsequently shown that, using this product, the quantum-covariance can be fully implemented. In particular, the quantum-covariant form of the Lorentz equation of electrodynamics, i.e.,

$$mc\frac{\acute{d}\mathcal{V}_\alpha}{ds} = \frac{e}{c}F_{\alpha\mu} \circ \mathcal{V}^\mu, \tag{23}$$

is equivalent, once expressed in terms of the wave function and integrated, with the Klein-Gordon equation in the presence of an electromagnetic field [21].

However, one of the disadvantage of using a product is that it depends on its two terms. In order to proceed further and to reach a full quantum+scale-covariance, we shall use another equivalent but simpler tool using instead operators. Let us define a velocity operator:

$$\widehat{\mathcal{V}^\mu} = \mathcal{V}^\mu + i\,\frac{\lambda}{2}\,\partial^\mu, \tag{24}$$

so that the quantum-covariant derivative writes:

$$\frac{\acute{d}}{ds} = \widehat{\mathcal{V}^\mu}\,\partial_\mu, \tag{25}$$

i.e., it now keeps its standard first order form. More generally, one defines the operator:

$$\frac{\widehat{\acute{d}\,g}}{ds} = \frac{\acute{d}\,g}{ds} + i\frac{\lambda}{2}\,\partial^\mu g\,\partial_\mu \tag{26}$$

which has the advantage to be defined only in terms of g. The covariant derivative of a product now writes:

$$\frac{\acute{d}\,(fg)}{ds} = \frac{\widehat{\acute{d}\,f}}{ds}\,g + \frac{\widehat{\acute{d}\,g}}{ds}\,f, \tag{27}$$

i.e., it keeps the form of the first order Leibniz rule.

Using this tool, equations (14) and (19) can be written under the compact form (in the free case):

$$\widehat{\mathcal{V}^\mu}\,\partial_\alpha\mathcal{V}_\mu = \widehat{\mathcal{V}^\mu}\,\partial_\mu\mathcal{V}_\alpha = 0. \tag{28}$$

4 Classical electrodynamics in scale relativity

4.1 Scale-covariant derivative

The theory of scale relativity allows one to get new insights about the nature of the electromagnetic field, of the electric charge, and about the physical meaning of gauge invariance [9, 10]. Consider an electron (or any other charged particle). In scale relativity, we identify the particle with a family of fractal trajectories, described as the geodesics of a nondifferentiable space-time. These trajectories are characterized by internal structures which are fractal, i.e. explicitly dependent on one (or several) scale variable(s) that we have named 'resolution' (by extension of the concept of resolution of a measurement apparatus).

Now consider anyone of these structures, lying at some (relative) resolution ε smaller that the Compton length of the particle (i.e. such that $\varepsilon < \lambda$) for a given relative position of the particle. In a displacement of the particle, the relativity of scales implies that the resolution at which this given structure appears in the new position will a priori be different from the initial one. Indeed, if the whole internal fractal structure of the electron was rigidly fixed, this would mean an absolute character of the scale space and a description of the fractal set of trajectories in terms of fractal rigid objects. Such a description would be clearly physically irrelevant.

Therefore we expect the occurrence of dilations of resolutions induced by translations. In other words, the scale variation includes a purely geometric increase which reads:

$$\delta\left(\ln\frac{\lambda}{\varepsilon}\right) = \frac{1}{q} A_\mu \, dx^\mu. \tag{29}$$

In this expression, the elementary dilation is written as $\delta \ln(\varepsilon/\lambda) = \delta\varepsilon/\varepsilon$: this is justified by the Gell-Mann-Levy method, from which the dilation operator is found to take the form $\tilde{D} = \varepsilon\partial/\partial\varepsilon = \partial/\partial \ln \varepsilon$. Since the elementary displacement in space-time δx^μ is a four-vector and since $\delta\varepsilon/\varepsilon$ is a scalar, one must introduce a four-index quantity A_μ from the application of Einstein's rule. The constant q measures the amplitude of the scale-motion coupling; it is subsequently identified with the active electric charge that intervenes in the potential. This form ensures that the dimensionality of A_μ be CL^{-1}, where C is the electric charge unit (e.g., $\varphi = q/r$ for a Coulomb potential).

In analogy with Einstein's construction of general relativity, we substract the new geometric effect from the total effect, in order to recover a purely inertial regime (see e.g. [1]). This can be expressed in terms of a scale-covariant derivative: we set $\eta = q \ln(\lambda/\varepsilon)$, and we define (note the sign correction to previous papers)

$$D_\mu \eta = \partial_\mu \eta - A_\mu. \tag{30}$$

4.2 Gauge invariance

Let us go on with the dilation 'field' A_μ. If one wants such a field to be physical, it must be defined whatever the initial scale from which we started. Moreover, the principle of scale relativity also means that, a scale being always relative to another reference scale (that defines the state of scale of the reference system) the scale can change for two equivalent reasons (which are in the end undistinguishable): either because of a scale change while the scale of reference is kept fixed, or because of a change of the reference scale itself. This is the same situation as in the case of motion laws: namely, it is equivalent to move an object 1 with respect to an object 0 (that serves as reference) or to make the reverse motion of object 0 relatively to object 1.

Therefore, starting from another relative scale $\varepsilon' = \varrho\varepsilon$ (we consider Galilean scale-relativity for the moment), where the scale ratio ϱ may be any function of coordinates, $\varrho = \varrho(x, y, z, t)$, we get

$$q\,\frac{\delta\varepsilon'}{\varepsilon'} = -A'_\mu \, \delta x^\mu, \tag{31}$$

so that we obtain:

$$A'_\mu = A_\mu + q\,\partial_\mu \ln \varrho(x, y, z, t). \tag{32}$$

Therefore the 4-vector A_μ depends on the relative "state of scale", or "scale velocity", $\ln \varrho = \ln(\varepsilon'/\varepsilon)$.

We have suggested [9, 10] to identify A_μ with an electromagnetic 4-potential and Eq. (32) with the gauge invariance relation for the electromagnetic field, that writes in the standard way:

$$A'_\mu = A_\mu + q\,\partial_\mu \chi(x, y, z, t), \tag{33}$$

where χ is usually considered as a function of coordinates devoid of physical meaning. This is no longer the case here, since it is now identified with a scale ratio $\chi = \ln \varrho$ between internal structures of the electron geodesics (at scales smaller than its Compton length). Our interpretation of the nature of the gauge function is compatible with its inobservability. Indeed, such a scale ratio is impossible to measure explicitly, since it would mean to make two measurements of two different relative scales smaller than the electron Compton length. But the very first measurement with resolution ε would change the state of the electron: namely, just after the measurement, its de Broglie length would become of order $\lambda_{\rm dB} \approx \varepsilon$ (see e.g. [8]), so that the second scale ε' would not be measured on the same electron. Therefore the ratio ϱ between the scales ε' and ε is destined to remain a virtual quantity. However, even whether it can not be directly measured, it has indirect consequences, so that the knowledge of its nature finally plays an important role: it allows one to demonstrate the quantization of the electron charge and to relate its value to that of its mass [9, 10, 18].

4.3 Definition of the electric charge

The fundamental new point of scale relativity with respect to the standard view is the fractal nature of space-time, i.e. its explicit dependence on the relative resolution scale, which is characterized by $\ln \varrho = \ln(\lambda/\varepsilon)$ in the simplified case of a global dilation. In other words, the space of positions and instants must be completed by a space of scales.

Consider the action S for the electron. In the framework of a space-time theory based on a relativity principle, as it is the case here, it should be given directly by the length invariant s, i.e., $dS = -mc\,ds$. This relation ensures that the stationary action principle $\delta \int dS = 0$ becomes identical with a geodesics (Fermat) principle $\delta \int ds = 0$. Now the fractality of the geodesical curves to which the electron is identified means that, while S is an invariant with respect to space-time changes of the coordinate system, it is however a function of the scale variable, $S = S(\eta)$, at scales smaller than λ.

Therefore we expect the action to be an explicit function of $\eta = q \ln \varrho$, so that its differential reads:

$$dS = \frac{\partial S}{\partial \eta}\, d\eta = \frac{\partial S}{\partial \eta}\,(D\eta + A_\mu dx^\mu), \tag{34}$$

so that we obtain:

$$\partial_\mu S = D_\mu S + \frac{\partial S}{\partial \eta}\, A_\mu. \tag{35}$$

What is the meaning of the derivative $-\partial S/\partial\eta$? Noether's theorem tells us that universal conservative quantities must emerge from the symmetries of the underlying space variables x^k; moreover, when considering the action as a function of coordinates at the upper limit of integration in the action integral, one finds that the conservative quantities are given by $p_k = -\partial_k S$. Now, space-time is completed in the scale-relativity framework by a scale space. Therefore, from the uniformity of the new scale variable $\ln \varrho$, a new conservative quantity can be constructed, so that the derivative of the action with respect to scale transformations actually provides us with a definition for the 'passive' electric charge [9], namely,

$$\frac{e}{c} = -\frac{\partial S}{\partial \eta} = -\frac{\partial S}{q\,\partial \ln \rho}. \tag{36}$$

In other words, the electric charge is defined in the new approach as the conservative quantity that arises from the uniformity of the scale space. The above choice is motivated by the expected symmetry of the 'active' and 'passive' charges in the final potential energy, and by the fact that the action has the dimensionality of an angular momentum $[\mathrm{ML^2T^{-1}}]$, while the squared charge dimensionality is $[\mathrm{ML^3T^{-2}}]$.

Finally, the known form of the coupling term in the action is now demonstrated, while it was merely postulated in the standard theory:

$$S_{\mathrm{pf}} = \int -\frac{e}{c} A_\mu dx^\mu. \tag{37}$$

4.4 Lorentz force deduced from a geodesics equation

4.4.1 Generalized invariant proper time

The field term of the action, as already recalled, is forced to be given by the square of the electromagnetic tensor. Therefore, the total action writes:

$$S = S_{\mathrm{p}} + S_{\mathrm{pf}} + S_{\mathrm{f}} = -\int mc\, ds - \int \frac{e}{c} A_\mu dx^\mu - \frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu} d\Omega. \tag{38}$$

Let us consider the two first terms of this action. In the new framework, they have both acquired a (geo)metric meaning. Indeed, the action is now equivalent to the space-time invariant, and the least action principle is therefore equivalent to a geodesics principle. The total elementary "length" (i.e., its proper time) of a fractal curve writes

$$ds_{\mathrm{tot}} = (g_{\mu\nu}dx^\mu dx^\nu)^{1/2} + \frac{e}{mc^2} A_\mu dx^\mu. \tag{39}$$

The new meaning of this expression in the scale-relativity / fractal space-time description is as follows: the length of a fractal trajectory can increase because its extremity has changed as a consequence of the motion of the particle (this is expressed by the first term, which is common to fractal and non-fractal trajectories), but also because of internal dilations (this is expressed by the second term). Finally the full length increase becomes distributed on the two terms: in the extreme cases, one can have a purely internal length increase that will have no counterpart in terms of space displacement and will therefore be equivalent to a potential energy, or one can also consider an unfolding of the fractal trajectory, in which this potential energy becomes manifested in terms of motion.

We shall therefore postulate that the real trajectories are given by an optimization of the full invariant proper length, i.e. by the generalized fractal geodesics equation,

$$\delta \int ds_{\mathrm{tot}} = 0. \tag{40}$$

4.4.2 Covariant derivative of a vector

Although we have recovered here the standard variational principle of classical electromagnetism, we shall nevertheless develop it again hereafter, since the new geometric interpretation will allow us to give a new meaning to the electromagnetic field: namely, we shall see that it can be identified with a fractal space-time connection that defines the covariant derivative of a vector. This amounts to follow in this new context Einstein's initial derivation of the motion equation in his relativistic theory of gravitation [2]. Indeed, in this derivation the definition of the covariant derivative and of the Christoffel symbols (subsequently identified with the gravitational field) directly proceed from the geodesics / least action principle.

The variation of the invariant reads:

$$\delta s_{\mathrm{tot}} = \int \left(\frac{dx_\mu d\delta x^\mu}{ds} + \frac{e}{mc^2} A_\mu d\delta x^\mu + \frac{e}{mc^2} \delta A_\nu dx^\nu \right) = 0. \tag{41}$$

After integration by parts and vanishing of the integrated terms (because the integral is varied for fixed values of the coordinates at its borns) it becomes:

$$\int \left(du_\mu \delta x^\mu + \frac{e}{mc^2} dA_\mu \delta x^\mu - \frac{e}{mc^2} \delta A_\nu dx^\nu \right) = 0. \tag{42}$$

Finally, since $\delta A_\nu = (\partial A_\nu / \partial x^\mu)\delta x^\mu$ and $dA_\mu = (\partial A_\mu / \partial x^\nu) dx^\nu$, we obtain:

$$\int \left\{ \frac{du_\mu}{ds} - \frac{e}{mc^2} \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) u^\nu \right\} \delta x^\mu ds = 0. \tag{43}$$

All the terms additional to the inertial ones have now a geometric origin. This leads us to define the covariant differential of the velocity as:

$$Du_\alpha = du_\alpha - \frac{e}{mc^2} F_{\alpha\mu} dx^\mu. \tag{44}$$

in terms of a 'connection'

$$F_{\alpha\mu} = \frac{\partial A_\mu}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\mu} \tag{45}$$

which is identified with the electromagnetic field. We are now able to define the covariant partial derivative of a resolution-dependent vector:

$$D_\mu B_\alpha = \partial_\mu B_\alpha + \frac{e}{mc^2} F_{\mu\alpha}. \tag{46}$$

In analogy with the construction of Einstein's covariant derivative in general relativity (see e.g. [1]), it amounts to subtract from the total variation of the vector the new variation of geometric origin, in order to let only its inertial part. While the geometric variation of a vector is a consequence of curvature in motion general relativity, and manifests itself in terms of gravitation, it is a consequence of fractality in scale relativity, and it manifests itself in terms of electromagnetism (and more generally in terms of gauge fields [22]).

4.4.3 Geodesics equation from covariance principle

Finally, applying a generalized strong covariance principle (that extends the covariance principle of general relativity), the Lorentz equation of electrodynamics is established in terms of a geodesics equation that keeps in terms of the covariant derivative the form of the Galilean equation of free motion, namely

$$\frac{Du_\alpha}{ds} = u^\mu D_\mu u_\alpha = \frac{du_\alpha}{ds} - \frac{e}{mc^2} F_{\alpha\mu} u^\mu = 0. \tag{47}$$

Its meaning is that a charged particle is actually locally in free fall in a space-time which is subjected to dilations and contractions at small scales (as a consequence of the presence of other charges). These local dilations and contractions of the internal structures of the particle trajectory manifest themselves at the macroscopic scales in terms of accelerations, which have been interpreted up to now as the effect of a force. In the new approch there is no longer any such force or field, but only manifestations of the expansion and contraction properties of the fractal geometry.

It should be remarked that, in the new scale-relativistic description the electromagnetic field is no longer separated from its source (the electron). The potential A^μ becomes a property of the 'electron' geometry itself, since it is identified with the ability of its 'internal' fractal structures to contract and dilate. Therefore the full 4-momentum expression of the electron has now two terms,

$$P^\mu = mcu^\mu + \frac{e}{c} A^\mu, \tag{48}$$

which are, as we have previously seen, respective manifestations of its motion and scale properties. The new interpretation of the motion of an 'electron' submitted to an 'electromagnetic field' is that its state of motion changes because there is a transfer of energy-momentum toward (or from) its internal fractal structures. One of the specificity of electromagnetism with respect to gravitation is easily accounted for by this view. Indeed, while gravitation is always attractive, electromagnetism can be attractive, repulsive or neutral (when acting on a uncharged particle). In the present interpretation, since electromagnetism is not directly related to the geometry but instead to its variations (expansion, contraction and staticity), one recovers its attractive, repulsive and neutral character.

4.4.4 Geodesics equation from energy equation

Before concluding this section, let us give another equivalent derivation of the Lorentz equation. Let us start from the energy equation:

$$ds^2 = dx^\mu dx_\mu \Rightarrow u^\mu u_\mu = 1, \tag{49}$$

and take its partial derivative:

$$\partial_\alpha (u^\mu u_\mu) = 0 \Rightarrow u^\mu \partial_\alpha u_\mu = 0. \tag{50}$$

As we shall see, this is nothing but the electrodynamics equation, under a form that is remarkable since there is no explicit presence of the field in it. In the free case it is indistinguishable from the free motion equation, since $\partial_\alpha u_\mu - \partial_\mu u_\alpha = 0$. In presence of an electromagnetic field this commutation relation becomes wrong. Indeed, since $mcu^\mu + (e/c)A^\mu = -\partial_\mu S$ is a gradient, we have:

$$\partial_\alpha u_\mu - \partial_\mu u_\alpha = \frac{e}{mc^2}\left(\partial_\mu A_\alpha - \partial_\alpha A_\mu\right). \tag{51}$$

so that

$$u^\mu \partial_\alpha u_\mu = u^\mu \left(\partial_\mu u_\alpha - \frac{e}{mc^2} F_{\alpha\mu}\right) = \frac{du_\alpha}{ds} - \frac{e}{mc^2} F_{\alpha\mu} u_\mu = 0. \tag{52}$$

Therefore the free-form equation $u^\mu \partial_\alpha u_\mu = 0$ is valid in both cases, without and with an electromagnetic field [21]. However, in the standard theory of electromagnetism in which the charges, the potential and their coupling are given from experiment without being derived from first principles, this is a mere result of the identification of the electromagnetic four-potential with an energy-momentum difference. In the scale-relativity framework, the nature of charges, of the field and the expression for their coupling are derived from a geometric description of space-time, so that this result acquires its full geometric and covariant meaning.

To conclude this section, recall that the variational principle applied on the two last terms of the full action including the field action, after generalization to the current of several charges, yields Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} j^\nu. \tag{53}$$

In conclusion of this section, the progress here respectively to the standard classical electromagnetic theory is that, instead of being independently constructed, the Lorentz force and the Maxwell equations are derived in the scale relativity theory as being both manifestations of the fractal geometry of space-time. Moreover, a new physical meaning has been given to the electric charge and to gauge transformations in this framework.

4.5 Link with Weyl-Dirac theory

Before generalizing the approach to non-Abelian gauge theories, let us notice that it shares some features with the Weyl-Dirac theory of electromagnetism [3, 4], but with new and essential differences. Namely, the Weyl theory considers scale transformations of the line element, $ds \to ds' = \rho\, ds$, but without specifying any fundamental cause for this dilation. The variation of ds should therefore exist at all scales, in contradiction with the observed invariance of the Compton-length of the electron (i.e., of its mass).

In the scale relativity proposal, the change of the line element comes from the fractal geometry of space-time, and it is therefore a consequence of the dilation of resolution. Moreover, the explicit effects of the dependence on resolutions is observable only below the fractal-nonfractal transition, which is identified in rest frame with the Compton scale of the particle. This solves the problem encountered in the Weyl theory and ensures the invariance of the observed electron mass in this theory.

Moreover, as we shall recall in the following, the interpretation of the gauge function as the logarithm of a scale ratio in the scale space allows one to obtain a fundamental new result. Indeed, in the framework of special scale relativity, scale ratios are limited because of the new status of the Planck length as a minimal scale, invariant under dilations. As a consequence the charge of the electron is quantized and its quantized value can be related to the ratio of its mass over the Planck mass.

5 Quantum electrodynamics

5.1 Introduction

The problem posed by the foundation of quantum electrodynamics in the framework of the theory of scale-relativity is far more difficult. Indeed, both the quantum properties and the electromagnetic properties are expected to be generated by the non-differentiable and fractal geometry of space-time in this framework. One should therefore combine the quantum-covariant derivative (that describes the induced effects of fractality and non-differentiability on motion) and the scale-covariant derivative (that describes the scale-motion coupling, i.e., coordinate-dependent resolutions) in terms of one unique covariant tool.

As in the classical case, our aim here is to succeed writing the Klein-Gordon and Dirac equations in the presence of an electromagnetic field in terms of a free geodesics equation.

5.2 QED-covariant derivative

Let us now recall how one can recover the standard QED quantum derivative in the scale-relativity approach. We consider again the generalized action introduced in section 4, which depends on motion and on scale variables. In the scale-relativistic quantum description, the 4-velocity is now complex, so that the action writes, $\mathcal{S} = \mathcal{S}(x^\mu, \mathcal{V}^\mu, \ln \varrho)$. This action gives the fundamental meaning of the wave function, namely, ψ is defined as:

$$\psi = e^{i\mathcal{S}/\hbar}. \tag{54}$$

Since the action is a complex number (and becomes a complex quaternion in the generalized case that leads to the demonstration of the Dirac equation [17]), this expression contains a phase and a modulus (that becomes in the end a square-root of probability density).

The decomposition of the action performed in the framework of the classical theory still holds and now becomes (for an electron of charge e):

$$d\mathcal{S} = -i\hbar \, d\ln\psi = -mc\mathcal{V}_\mu dx^\mu - \frac{e}{c}A_\mu dx^\mu. \tag{55}$$

Equation (55) allows one to define a generalized complex energy-momentum,

$$\mathcal{P}^\mu = mc\mathcal{V}^\mu + \frac{e}{c}\, A^\mu. \tag{56}$$

This leads to a new expression for the relation between the complex velocity and the wave function:

$$\mathcal{V}_\mu = i\lambda\, \partial_\mu(\ln\psi) - \frac{e}{mc^2}A_\mu, \tag{57}$$

where $\lambda = \hbar/mc$ is the Compton length of the electron.

Therefore we can generalize the identity $\mathcal{V}_\mu = i\lambda\, \partial_\mu(\ln\psi)$ to its covariant form by introducing a covariant derivative:

$$\mathcal{V}_\mu = i\lambda\, D_\mu(\ln\psi). \tag{58}$$

We recognize in this derivative the standard QED-covariant derivative operator when it is acting on the wave function ψ:

$$-i\hbar D_\mu = -i\hbar\partial_\mu + \frac{e}{c}\, A_\mu, \tag{59}$$

since we can write Eq. (57) as $mc\mathcal{V}_\mu\psi = [i\hbar\partial_\mu - (e/c)A_\mu]\, \psi$.

This provides one with an understanding from first principles of the nature and origin of the QED covariant derivative, while it was merely set as a rule devoid of geometric meaning in the standard quantum field theory.

This covariant derivative is directly related to the one introduced in the classical framework. Indeed, the classical covariant derivative was written $D_\mu = \partial_\mu + (1/q)A_\mu$ acting on ϱ, while $\psi = \psi_0\, \exp[(i/\hbar)(eq/c)\ln\varrho]$. We therefore recover expression (59) acting on ψ.

5.3 Electromagnetic KG equation from a geodesics equation

We are now able to combine the quantum-covariant and the scale-covariant derivatives in terms of a common tool. Recall that the quantum-covariance has been fully implemented by the use of a covariant complex velocity operator $\widehat{\mathcal{V}^\mu} = \mathcal{V}^\mu + i(\lambda/2)\partial^\mu$ (Sec. 3.6). Indeed, the quantum-covariant derivative reads in its term $\acute{d}/ds = \widehat{\mathcal{V}^\mu}\, \partial_\mu$.

Then the classical electromagnetic field has been constructed thanks to a scale-covariant derivative D_μ that manifests the expansion-contraction properties of the fractal space-time, such that the equation of motion reads $Du_\alpha/ds = u^\mu D_\mu u_\alpha = 0$.

One can therefore combine both tools and define a scale + quantum covariant derivative:

$$\frac{\widetilde{D}}{ds} = \widehat{\mathcal{V}^\mu}\; D_\mu \tag{60}$$

and we can finally write a free-like, covariant, geodesics equation:

$$\frac{\widetilde{D}}{ds}\mathcal{V}_\alpha = 0. \tag{61}$$

We shall now prove that this extremely simple, free-form equation gives, after integration, the Klein-Gordon equation in the presence of an electromagnetic field. In other words, this means that it contains all the quantum terms and all the field terms, which are both generated through the double scale-covariance (that manifests the non-differentiability and the fractality).

As a first step, let us show that it allows one to obtain a quantum-covariant form of the Lorentz equation of dynamics. By successively developping the covariant derivatives, it becomes

$$\frac{\widetilde{D}}{ds}\mathcal{V}_\alpha = \widehat{\mathcal{V}^\mu}\, D_\mu \mathcal{V}_\alpha = \widehat{\mathcal{V}^\mu}\left(\partial_\mu \mathcal{V}_\alpha + \frac{e}{mc^2}F_{\mu\alpha}\right) = 0, \tag{62}$$

and since $\widehat{\mathcal{V}^\mu}\, \partial_\mu \mathcal{V}_\alpha = \acute{d}\,\mathcal{V}_\alpha / ds$, we obtain:

$$mc\,\frac{\acute{d}}{ds}\mathcal{V}_\alpha = \frac{e}{c}\,\widehat{\mathcal{V}^\mu}\,F_{\alpha\mu}, \tag{63}$$

which has exactly the form of the Lorentz equation, although this is a quantum equation whose integral is the Klein-Gordon equation with electromagnetic field. This equation is equivalent to that written by Pissondes in the scale-relativity framework by using a symmetric product [21]. But in this previous work there was no justification of the existence of an electromagnetic field, which was included by using the standard QED covariant derivative. The additional point here is that the theory generates both the field and the quantum behavior.

Let us go on from Eq. (62) and now develop the complex velocity operator. We obtain:

$$\left(\mathcal{V}^\mu + i\frac{\lambda}{2}\partial^\mu\right)\,\left(\partial_\mu \mathcal{V}_\alpha + \frac{e}{mc^2}\,F_{\mu\alpha}\right) = 0. \tag{64}$$

While $\partial_\mu \mathcal{V}_\alpha - \partial_\alpha \mathcal{V}_\mu = 0$ in the free case, it no longer commutes in the presence of an electromagnetic field. As in the classical situation it becomes

$$\partial_\mu \mathcal{V}_\alpha - \partial_\alpha \mathcal{V}_\mu = -\frac{e}{mc^2}F_{\mu\alpha}. \tag{65}$$

We therefore obtain a form of the equation in which the indices are exchanged with respect to the free case:

$$\left(\mathcal{V}^\mu + i\frac{\lambda}{2}\partial^\mu\right)\partial_\alpha \mathcal{V}_\mu = \widehat{\mathcal{V}^\mu}\,\partial_\alpha \mathcal{V}_\mu = 0. \tag{66}$$

We now replace $\mathcal{V}^\mu$ by its covariant form and we obtain:

$$\left(i\lambda\,\partial^\mu(\ln\psi) - \frac{e}{mc^2}A^\mu + i\frac{\lambda}{2}\partial^\mu\right)\partial_\alpha\left(i\lambda\,\partial_\mu(\ln\psi) - \frac{e}{mc^2}A_\mu\right) = 0. \tag{67}$$

After integration, this equation becomes the Klein-Gordon equation for a particle in an electromagnetic field:

$$\left(i\hbar\partial_\mu - \frac{e}{c}A_\mu\right)\left(i\hbar\partial^\mu - \frac{e}{c}A^\mu\right)\psi = m^2c^2\psi. \tag{68}$$

The expression (66) of the motion equation can also be obtained by differentiating the energy equation [21]:

$$\partial_\alpha(\mathcal{V}^\mu\mathcal{V}_\mu + i\lambda\partial^\mu\mathcal{V}_\mu) = 0 \Rightarrow \widehat{\mathcal{V}^\mu}\ \partial_\alpha\mathcal{V}_\mu = 0. \tag{69}$$

This equation is valid without and with electromagnetic field, in the same way as its classical equivalent, $u^\mu\partial_\alpha u_\mu = 0$.

The Dirac equation in an electromagnetic field can then be derived by the same method that has yielded the free Dirac equation from the Klein-Gordon one, i.e. by the use of biquaternions (that describe the new two-valuedness which is a consequence of the symmetry breaking under the reflection $dx^\mu \leftrightarrow -dx^\mu$, issued from non-differentiability).

5.4 Nature of the electric charge (quantum theory)

In a gauge transformation $A'_\mu = A_\mu + e\partial_\mu\chi$ the wave function of an electron of charge e becomes:

$$\psi' = \psi\ \exp\left\{\frac{i}{\hbar} \times \frac{e}{c} \times e\chi\right\}. \tag{70}$$

We have reinterpreted in the previous sections the gauge transformation as a scale transformation of resolution, $\varepsilon \to \varepsilon'$, yielding an identification of the gauge function with a scale ratio, $\chi = \ln\varrho = \ln(\varepsilon/\varepsilon')$, which is a function of space-time coordinates. In such an interpretation, the specific property that characterizes a charged particle is the explicit scale-dependence on resolution of its action, then of its wave function. The net result is that the electron wave function writes

$$\psi' = \psi \exp\left\{i\frac{e^2}{\hbar c}\ln\varrho\right\}. \tag{71}$$

Since, by definition (in the system of units where the permittivity of vacuum is 1),

$$e^2 = 4\pi\alpha\hbar c, \tag{72}$$

where α is the fine structure constant, equation (71) becomes

$$\psi' = \psi\, e^{i4\pi\alpha\ln\varrho}. \tag{73}$$

This result supports the previous solution brought to the problem of the nature of the electric charge in the classical theory. Indeed, considering now the wave function of the electron as an explicitly resolution-dependent function, we can write the scale differential equation of which it is solution as:

$$-i\hbar\frac{\partial\psi}{\partial\left(\frac{e}{c}\ln\varrho\right)} = e\psi. \tag{74}$$

We recognize in $\tilde{D} = -i(\hbar c/e)\partial/\partial \ln \varrho$ a dilatation operator similar to that introduced in Section 3. Equation (74) can then be read as an eigenvalue equation issued from an extension of the correspondence principle (but here, demonstrated),

$$\tilde{D}\psi = e\psi. \tag{75}$$

This is the quantum expression of the above classical suggestion, according to which the electric charge is understood as the conservative quantity that comes from the new scale symmetry, namely, from the uniformity of the resolution variable $\ln \varepsilon$.

5.5 Charge quantization and mass-coupling relations

While the results of the scale relativity theory described in the previous sections mainly deal with a new interpretation of the nature of the electromagnetic field, of the electric charge and of gauge invariance, we now arrive at one of the main consequences of this approach: as we shall see, it allows one to establish the universality of the quantization of charges (for any gauge field) and to theoretically predict the existence of fundamental relations between mass scales and coupling constants.

In the previous section, we have recalled our suggestion [9, 10, 18] to elucidate the nature of the electric charge as being the eigenvalue of the dilation operator corresponding to resolution transformations (internal to the geodesics, identified with the 'particle'). We have written the wave function of a charged particle under the form Eq. (74).

Let us now consider in more detail the nature of the scale factor $\ln \varrho$ in this expression. This factor describes the ratio of two relative resolution scales ε and ε' that correspond to structures of the fractal geodesical trajectories that we identify with the electron. However the electron is not structured at all scales, but only at scales smaller than its Compton length $\lambda = \hbar/m_e c$. We can therefore take this upper limit as one of the two scales and write:

$$\psi' = \exp\left\{ i \; 4\pi\alpha \; \ln\left(\frac{\lambda}{\varepsilon}\right)\right\} \; \psi. \tag{76}$$

In the case of Galilean scale-relativity, such a relation leads to no new result, since ε can go to zero, so that $\ln(\lambda/\varepsilon)$ is unlimited. But in the framework of special scale-relativity, scale laws take a log-Lorentzian form below the scale λ (see Section 2). The Planck length $\lambda_{\mathbb{P}}$ becomes a minimal, unreachable scale, invariant under dilations, so that $\ln(\lambda/\varepsilon)$ becomes limited by $\mathbb{C} = \ln(\lambda/\lambda_{\mathbb{P}})$. This implies a quantization of the charge which amounts to the relation $4\pi\alpha\mathbb{C} = 2k\pi$, i.e.

$$\alpha\,\mathbb{C} = \frac{1}{2}k, \tag{77}$$

where k is integer. Since $\mathbb{C} = \ln(\lambda/\lambda_{\mathbb{P}})$ and is equal to $\ln(m_{\mathbb{P}}/m_e)$ for the electron (where $m_{\mathbb{P}}$ is the Planck mass), equation (77) amounts to a general relation between mass scales and coupling constants.

In order to explicitly apply such a relation to the electron, we must account for the fact that we now know from the electroweak theory that the electric charge is only a residual of a more general, high energy electroweak coupling. This coupling can be defined from the U(1) and SU(2) couplings as:

$$\alpha_0^{-1} = \frac{3}{8}\alpha_2^{-1} + \frac{5}{8}\alpha_1^{-1}. \tag{78}$$

It is such that $\alpha_0 = \alpha_1 = \alpha_2$ at unification scale and it is related to the fine structure constant at Z scale by the relation $\alpha = 3\alpha_0/8$. This means that, because the weak gauge bosons acquire mass through the Higgs mechanism, the interaction becomes transported at low energy only by the residual null mass photon. As a consequence the amplitude of the electromagnetic force abruptly falls by a factor 3/8 at the WZ scale. Therefore we have suggested that α_0 instead of α must be used in Eq. (77) for relating the electron mass to its charge.

Finally, disregarding as a first step threshold effects (that occur at the Compton scale), we get a mass-charge relation for the electron [9, 10]:

$$\ln \frac{m_P}{m_e} = \frac{3}{8}\alpha^{-1}. \tag{79}$$

The existence of such a relation between the mass and the charge of the electron is supported by the experimental data. Indeed, using the known experimental values, the two members of this equation agree to 0.2 %: $\mathbb{C}_e = \ln(m_{\mathbb{P}}/m_e) = 51.528 \pm 0.001$ while $(3/8)\alpha^{-1} = 51.388$. The agreement is made even better if one accounts for the fact that the measured fine structure constant (at Bohr scale) differs from the limit of its asymptotic behavior (that includes radiative corrections). One finds that the asymptotic inverse running coupling at the scale where the asymptotic running mass reaches the observed mass m_e is $\alpha_0^{-1}\{r(m = m_e)\} = 51.521$, which lies within 10^{-4} of the value of $\mathbb{C}_e$.

6 Generalization to non-Abelian gauge fields

6.1 Introduction

Let us now recall how one can generalize the electromagnetic description to non-Abelian gauge theories [22]. We consider that the internal fractal structures of the "particle" (i.e. of the family of geodesics of a non-differentiable space-time) are now described in terms of several scale variables $\eta_{\alpha\beta\ldots}(x, y, z, t)$, written for simplicity in λ units and that generalize the single resolution variable ε. We assume that the various indices can be gathered into one common index: we therefore write the scale variables under the simplified form η_α ($\alpha = 0$ to N).

In the simplest case, $\eta_\alpha = \varepsilon_\alpha$, where ε_α is the resolution of the space-time coordinate X_α ($\alpha = 1$ to 4). However, other situations can be considered. In this article, we present the development of non-Abelian gauge theory in terms of the more general scale transformations, since our present aim is mainly to relate in a general way the scale relativistic tools to the standard description of current gauge theories. Only the peculiar example of rotations in scale space, which we identify to the isospin transformation group, is presented in section

6.5 below. Other sets of transformations will be studied in forthcoming works, where the scale variables will be given other precise definitions.

6.2 General scale transformations

Let us now consider infinitesimal scale transformations. The transformation law on the η_α can be written in a linear way as

$$\eta'_\alpha = \eta_\alpha + \delta\eta_\alpha = (\delta_{\alpha\beta} + \delta\theta_{\alpha\beta})\, \eta^\beta, \tag{80}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol, or equivalently,

$$\delta\eta_\alpha = \delta\theta_{\alpha\beta}\, \eta^\beta. \tag{81}$$

Let us now assume that the η_α's are functions of the standard space-time coordinates. This leads us to generalize the scale-covariant derivative previously defined in the electromagnetic case as follows: the total variation of the resolution variables becomes the sum of the inertial one, described by the covariant derivative, and of the new geometric contribution, namely,

$$d\eta_\alpha = D\eta_\alpha - \eta^\beta \delta\theta_{\alpha\beta} = D\eta_\alpha - \eta^\beta W^\mu_{\alpha\beta}\, dx_\mu. \tag{82}$$

Recall that in the Abelian case, which corresponds to a unique global dilation, this expression can be simplified since $d\eta/\eta = d\ln\eta = d\chi$.

Therefore, in this new situation we are led to introduce "gauge fields" $W^\mu_{\alpha\beta}$ that enter naturally in the geometrical frame of Eq. (82). These fields are linked to the scale transformations as follows,

$$\delta\theta_{\alpha\beta} = W^\mu_{\alpha\beta}\, dx_\mu. \tag{83}$$

One should remain cautious about this expression and keep in mind that these fields find their origin in a covariant derivative process and are therefore not gradients (this is expressed by the use of a difference sign $\delta\theta_{\alpha\beta}$ instead of $d\theta_{\alpha\beta}$). They formalize the coupling between displacements in space-time and dilations/contractions of the scale variables and play in Eq. (82) a role analogous to the one played in General Relativity by the Christoffel symbols. It is also, once again, important to notice that the $W^\mu_{\alpha\beta}$ introduced at this level of the analysis do not include charges. They are a function only of the space and time coordinates. This is a necessary choice because our method generates, as we shall see again, not only the fields but also the charges from respectively the scale transformations and the scale symmetries of the dynamical fractal space-time.

6.3 Generalization of bi-spinors: multiplets

After having written the transformation law of the basic variables (the η_α), we are now led to describe how various physical quantities transform under the η_α transformations. These new laws of transformation are expected to depend on the nature of the objects to transform (e.g. vectors, tensors, spinors, etc...), which implies to jump to group representations.

In the case where the particle is a spin 1/2 fermion, the relation between the velocity and the spinor fields reads [17]

$$\mathcal{V}_\mu = i\lambda\, \psi^{-1}\partial_\mu\psi, \tag{84}$$

where $\mathcal{V}_\mu$ and ψ are complex quaternions and the constant $\lambda = \hbar/mc$ is the Compton length of the particle in rest frame.

However, bispinors are not yet a general enough description for fermions subjected to a general gauge field. Indeed, we consider here a generalized group of transformations which therefore involves generalized charges. As a consequence of these new charges, the very nature of the fermions is expected to become more complicated. Experiments have indeed shown that new degrees of freedom must be added in order to represent the weak isospin, hypercharge and color. In order to account in a general way for this more complicated description, we shall simply introduce multiplets ψ_k, where each component is a Dirac bispinor. Therefore, as already remarked in previous publications [10], when the scale variables become multiplets, the same is true of the charges. As we shall see in what follows, in the present approach it is at the level of the construction of the charges that the group generators enter.

In this case the multi-valued velocity becomes a bi-quaternionic matrix,

$$\mathcal{V}^\mu_{jk} = i\lambda\, \psi_j^{-1}\partial^\mu\psi_k. \tag{85}$$

Therefore the action becomes also a tensorial two-index quantity,

$$dS_{jk} = dS_{jk}(x^\mu, \mathcal{V}^\mu_{jk}, \eta_\alpha). \tag{86}$$

In the absence of a field, it is linked to the generalized velocity (and therefore to the spinor multiplet) by the relation,

$$\partial^\mu S_{jk} = -mc\, \mathcal{V}^\mu_{jk} = -i\hbar\, \psi_j^{-1}\partial^\mu\psi_k. \tag{87}$$

6.4 General definition of the charges

Now, in the presence of a field (i.e. when the second-order effects of the fractal geometry appearing in the right hand side of Eq. (82) are included), using the complete expression for $\partial^\mu\eta_\alpha$,

$$\partial^\mu\eta_\alpha = D^\mu\eta_\alpha - W^\mu_{\alpha\beta}\,\eta^\beta, \tag{88}$$

we are led to write a relation that generalizes Eq. (55) to the non-Abelian case,

$$\partial^\mu S_{jk} = \frac{\partial S_{jk}}{\partial\eta_\alpha}\,\partial^\mu\eta_\alpha = \frac{\partial S_{jk}}{\partial\eta_\alpha}\,(D^\mu\eta_\alpha - W^\mu_{\alpha\beta}\,\eta^\beta). \tag{89}$$

Thus we obtain

$$\partial^\mu S_{jk} = D^\mu S_{jk} - \eta^\beta\,\frac{\partial S_{jk}}{\partial\eta_\alpha}\,W^\mu_{\alpha\beta}. \tag{90}$$

We are finally led to define a general group of scale transformations whose generators are :

$$T^{\alpha\beta} = \eta^\beta \partial^\alpha, \tag{91}$$

(where we use the compact notation $\partial^\alpha = \partial/\partial\eta_\alpha$), yielding the generalized charges

$$\frac{\tilde{g}}{c}\, t^{\alpha\beta}_{jk} = \eta^\beta \,\frac{\partial S_{jk}}{\partial \eta_\alpha}. \tag{92}$$

This group is submitted to an unitarity condition, since when it is applied on the wavefunctions, $\psi\psi^\dagger$ must be conserved.

The main point to be stressed here is the natural appearance of the $t^{\alpha\beta}_{jk}$ in the very definition of the charges as a mere consequence of the partial derivation. To sum up the above demonstration (omitting the indices), we have written $\partial_\mu S$ as the product $(\partial S/\partial\eta)\times\partial_\mu\eta$. The second term reads ηW_μ, and the product therefore recombines itself in two terms, $\eta(\partial S/\partial\eta)$ which identifies with the charges and W_μ which is the field. As a consequence the $t^{\alpha\beta}_{jk}$ coefficients do not need to appear in the initial definition of the general scale transformation, which ensures its universality and allows the geometric interpretation.

6.5 Rotations in scale-space

In order to enlight the meaning of the new definition we have obtained for the charges, we consider in the present section a sub-sample of the possible scale transformations on intrinsic resolutions: namely, those that can be described in terms of rotations. They constitute the antisymmetric part of the gauge group. In this case the infinitesimal transformation is such that

$$\delta\theta_{\alpha\beta} = -\delta\theta_{\beta\alpha} \Rightarrow W^\mu_{\alpha\beta} = -W^\mu_{\beta\alpha}. \tag{93}$$

Therefore, reversing the indices in Eq.(90), we may write

$$\partial_\mu S_{jk} = D_\mu S_{jk} - \eta^\alpha \,\frac{\partial S_{jk}}{\partial \eta_\beta}\, W^\mu_{\beta\alpha}. \tag{94}$$

Taking the half-sum of Eqs.(90) and (94) we finally obtain

$$\partial_\mu S_{jk} = D_\mu S_{jk} - \frac{1}{2}\left(\eta^\beta \,\frac{\partial S_{jk}}{\partial \eta_\alpha} - \eta^\alpha \,\frac{\partial S_{jk}}{\partial \eta_\beta}\right) W^\mu_{\alpha\beta}. \tag{95}$$

This leads to another definition of the charges,

$$\frac{\tilde{g}}{c}\, t^{\alpha\beta}_{jk} = \frac{\partial S_{jk}}{\partial \theta_{\alpha\beta}} = \frac{1}{2}\left(\eta^\beta \,\frac{\partial S_{jk}}{\partial \eta_\alpha} - \eta^\alpha \,\frac{\partial S_{jk}}{\partial \eta_\beta}\right). \tag{96}$$

We recognize here the form of the definition of an angular momentum from the derivative of the action, i.e. of the conservative quantity that finds its origin in the isotropy of space; but the space under consideration here is the "scale-space". Therefore the charges of the gauge fields are identified, in this interpretation, with "scale-angular momenta".

The subgroup of transformations corresponding to these generalized charges is, in three dimensions (i.e. for the three space resolutions), a SO(3) group related to a SU(2) group by the homomorphism which associates the same rotation to two distinct 2×2 unitary matrices of opposite sign. We are therefore naturally led to define a "scale-spin", which we propose to identify to the simplest non-Abelian charge in the current standard model: the weak isospin.

Coupling this SU(2) representation of the rotations in "scale-space" to the U(1) representation of the global scale dilations (that describes the electromagnetism process), we are therefore able to give a geometric physical meaning to the transformation group corresponding to the U(1)×SU(2) representation of the standard electroweak theory.

When it is applied to the four-dimensional scale-space (which is Euclidean since the space and time fractal fluctuations add quadratically), the subgroup of rotations of the gauge group is a SO(4) group, which contains the U(1) × SU(2) group of the standard electroweak theory. This result justifies the recent proposal [24, 25] of extension of the electroweak theory using this group that lead to a theoretical prediction of the Higgs boson mass $m_H = \sqrt{2}\, m_w \approx 114$ GeV (see Sec. 6.6).

6.6 Recovering Yang-Mills theories

For the developments to follow, we shall simplify the notation and use only one index $a = (\alpha, \beta)$ for the scale transformations, i.e. running on the gauge group parameters, now written θ_a. For example, in three dimensions this means that we replace the three rotations $\theta_{23}, \theta_{31}, \theta_{12}$ respectively by $\theta_1, \theta_2, \theta_3$. We obtain the following more compact form for the complete action

$$dS_{jk} = \left(D_\mu S_{jk} - \frac{\tilde{g}}{c}\, t^a_{jk}\, W_{a\mu} \right) dx^\mu, \tag{97}$$

and therefore

$$D^\mu S_{jk} = -i\hbar\, \psi_j^{-1} D^\mu \psi_k = -i\hbar\, \psi_j^{-1} \partial^\mu \psi_k + \frac{\tilde{g}}{c}\, t^a_{jk}\, W^\mu_a. \tag{98}$$

The previous equations used new concepts that are specific of the scale relativity approach, namely the scale variables η_α, the bi-quaternionic velocity matrix $\mathcal{V}^\mu_{jk}$ and its associated action S_{jk}. We shall now show that the standard basic concepts of quantum field theories, namely the fermionic field ψ, the bosonic field W^μ_a, the charge g, the gauge group generators t^a_{jk} and the gauge-covariant derivative D_μ can be derived from these new tools.

Let us indeed show that we are able to recover the various relations of standard non-Abelian gauge theories. From the previous equation, we first recover the standard form for the covariant partial derivative, now acting on the wavefunction multiplets,

$$D^\mu \psi_k = \partial^\mu \psi_k + i\, \frac{\tilde{g}}{\hbar c}\, t^{ja}_k\, W^\mu_a\, \psi_j. \tag{99}$$

The ψ_i do not commute together since they are bi-quaternionic quantities, but this is the case neither of t^{ja}_k nor of W^μ_a, so that ψ_j can be put to the right as in the standard way of writing; from the multiplet point of view (index j), we simply exchange the lines and the columns.

Now introducing a dimensionless coupling constant α_g and a dimensionless charge g, such that

$$g^2 = 4\pi\alpha_g = \frac{\tilde{g}^2}{\hbar c}, \tag{100}$$

and redefining the dimensionality of the gauge field (namely, we replace $W_a^\mu/\sqrt{\hbar c}$ by W_a^μ), the covariant derivative may be more simply written under its standard form,

$$D^\mu \psi_j = \partial^\mu \psi_j + i\, g\, t_j^{ka}\, W_a^\mu\, \psi_k. \tag{101}$$

In the simplified case of a fermion singlet, it reads

$$D^\mu = \partial^\mu + i\, g\, t^a\, W_a^\mu. \tag{102}$$

Let us now derive the laws of gauge transformation for the fermion field. Consider a transformation θ_a of the scale variables. As we shall now see, the θ_a can be identified with the standard parameters of a non-Abelian gauge transformation. Indeed, using the above remark about the exchange of lines and columns, Eq. (87) becomes

$$-i\hbar\partial^\mu \psi_j = \partial_\mu S_j^k\, \psi_k. \tag{103}$$

and then we obtain Eq. (101), from which we recover the standard form for the transformed fermion multiplet in the case of an infinitesimal gauge transformation $\delta\theta_a$,

$$\psi'_j = (\delta_j^k - ig\, t_j^{ka}\, \delta\theta_a)\psi_k. \tag{104}$$

We now have at our disposal all the tools of quantum gauge theories. The subsequent developments are standard ones in terms of these tools. Namely, one introduces the commutator of the matrices t_a (which have a priori no reason to commute),

$$t_a t_b - t_b t_a = f_{ab}^c\, t_c. \tag{105}$$

Therefore the t_a are identified with the generators of the gauge group and the f_{ab}^c with the structure constants of its associated Lie algebra. The non-commutativity finally implies the appearance of an additional term in the boson field law of the gauge transformation. One finds, still in the case of an infinitesimal gauge transformation $\delta\theta^a$, that it transforms according to

$$\delta W_\mu^a = \partial_\mu \delta\theta^a + g\, f_{bc}^a\, \delta\theta^b\, W_\mu^c. \tag{106}$$

We recognize here once again the standard transformations of non-Abelian gauge theories, which is now derived from the basic transformations on the η_a of Eq. (82).

Under these gauge transformations, the fermion multiplet and the boson field transform in such a way that the Lagrangian $\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$ (that contains the fermion and fermion-boson coupling thanks to the covariant derivative) remains invariant. It is worth stressing here that, in the scale relativity framework, this form of the Lagrangian is derived: the free Dirac form directly follows from the free equation of motion established in [17] and the covariant derivative contribution yielding the terms for the coupling to the fields has been stated here where it is given by Eq. (101). The gauge field self-coupling term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, which is defined as the simplest gauge invariant scalar that can be added to the Lagrangian, yields the standard Yang-Mills equations. We are therefore provided with a consistent gauge theory obtained as a consequence of scale symmetries issued from a geometric, non-differentiable and fractal space-time description.

6.7 Generalized mass coupling relations

In [10, 23], we also attempted to apply the mass-charge relation to the SU(2) coupling α_2. We found that the relation

$$3\,\alpha_{2Z}\,\mathbb{C}_Z = 4 \tag{107}$$

was precisely achieved at the Z scale. The origin of the factor 3 in this relation relies on the generalization of scale (i.e. gauge) transformations to dilations which are no longer global: one can instead consider different and independent dilations on the internal resolutions corresponding to the various coordinates. The group SU(2) corresponds to rotations in a 3-dimensional scale-space. Therefore the phase term in a fermion field writes:

$$\alpha_2\,\ln(\frac{\varepsilon_x}{\lambda}) + \alpha_2\,\ln(\frac{\varepsilon_y}{\lambda}) + \alpha_2\,\ln(\frac{\varepsilon_z}{\lambda}) < 3\,\alpha_2\,\ln(\frac{l_P}{\lambda}), \tag{108}$$

since the same coupling applies to the three variables, and since all three resolutions are limited toward small scales by the Planck scale. From Eq. (107) we expect a value $\alpha_{2Z}^{-1} = 29.8169 \pm 0.0002$. The present precise experimental value is [27]:

$$\alpha_{2Z}^{-1} = \alpha_Z^{-1} \times \hat{s}_Z^2 = 29.802 \pm 0.027, \tag{109}$$

which lies within 0.6 σ of the theoretical prediction.

More generally, the universality of mass-coupling relations for any gauge field implies that coupling ratios at Z scale should be integers. Present values ($\alpha_{1Z}^{-1} = 59.471 \pm 0.026$; $\alpha_{2Z}^{-1} = 29.802 \pm 0.027$; $\alpha_{3Z}^{-1} = 8.47 \pm 0.14$) [27] support this theoretical prediction within a 2σ confidence interval, since one obtains:

$$\left(\frac{\alpha_3}{\alpha_1}\right)_Z = 7.02 \pm 0.12 \quad ; \quad \left(\frac{\alpha_2}{\alpha_1}\right)_Z = 1.996 \pm 0.002. \tag{110}$$

6.8 Higgs boson mass

The framework of generalized scale-relativity provides one with possibilities to suggest a new version of the electroweak theory and, as a consequence, to make a theoretical prediction of the value of the Higgs boson mass [24, 25]. The (summarized) argument is as follows.

In today's electroweak scheme, the Higgs boson is considered to be separated from the electroweak field. Moreover, a more complete unification is mainly seeked in terms of attempts of "grand" unifications with the strong field. However, one may wonder whether, maybe in terms of an effective theory at intermediate energy, one could achieve a more tightly unified purely electroweak theory. Recall indeed that in the present standard model, the weak and electromagnetic fields are mixed, but there remains four free parameters, which can e.g. be taken to be the Higgs boson mass, the vacuum expectation value of the Higgs field and the Z and W masses. In the attempt sketched out hereafter, the Higgs field is assumed to be a part of the total field, so that only two free parameters would be left. As a consequence, the Higgs boson mass and the W/Z mass ratio could be derived in such a model.

Recall that the structure of the present electroweak boson content is as follows. There is a SU(2) gauge field, involving three fields of null mass (i.e. $2 \times 3 = 6$ degrees of freedom), a U(1) null mass field (2 d.f.) and a Higgs boson complex doublet (4 d.f.), which makes 12 degrees of freedom in all. Through the Glashow-Salam-Weinberg mechanism, 3 of the 4 components of the Higgs doublet become longitudinal components of the weak field which therefore acquires mass ($3 \times 3 = 9$ d.f.), while the photon remains massless (2 d.f.), so that there remains a Higgs scalar which is nowadays experimentally searched (1 d.f.).

Let us now consider four independant scale transformations on the four space-time resolutions, i.e., $(\varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_t)$, considered as intrinsic to the description and being now variable with space-time coordinates. This means that the scale space (i.e., here the gauge space) is at least four-dimensional (but note that this is not the final word on the subject, since this does not yet include the fifth "djinn" dimension δ that appears in special and general scale-relativity). Moreover, the mixing relation between the B [U(1)] and W_3 [SU(2)] fields may also be interpreted as a rotation in the full gauge (scale) space. Therefore we expect the appearance of a 6 component antisymmetric tensor field (linked to the rotations in this space), corresponding in the simplest case to a SO(4) group. Such a zero mass field would yield 12 degrees of freedom by itself alone, so that it is able to include the electromagnetic and weak fields, but also the residual Higgs field.

We shall tentatively explore the possibility that the Higgs boson appears as a separated scalar only as a low energy approximation, while in the new framework it would be one of the components of the unified field. An analogy can be made with space-time physics, in which energy appears as scalar at low velocity, while it is ultimately a component of the energy-momentum four-vector. Here we consider a similar situation, but in the scale-space (i.e. the gauge space): the large scale theory would be a Galilean approximation of special scale-relativity, involving a spontaneous symmetry breaking of the full gauge group as a mere result of the change of scale.

Such an attempt is supported by the form of the electroweak Lagrangian (we adopt Aitchison's notations [26]). Its Higgs scalar boson part writes (in terms of the residual massive scalar σ):

$$L_H = \frac{1}{2}\, \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2}\, m_H^2 \sigma^2 - \frac{1}{8} \lambda^2 \sigma^4. \tag{111}$$

The vacuum expectation value v of the Higgs field is computed from the square (mass term) and quartic term, so that the Higgs mass is related to v and λ as:

$$m_H = \sqrt{2}\, v\, \lambda. \tag{112}$$

A prediction of the constant λ would therefore lead to a prediction of the Higgs mass. Now, a non-Abelian field writes in terms of its potential :

$$F^{\alpha\mu\nu} = \partial^\mu W^{\alpha\nu} - \partial^\nu W^{\alpha\mu} - g\, c^\alpha_{\beta\gamma} W^{\beta\mu} W^{\gamma\nu}, \tag{113}$$

where g is the (now unique) charge and $c^\alpha_{\beta\gamma}$ the structure coefficients of the Lie algebra associated to the gauge group. Its Lagrangian writes:

$$L_W = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \tag{114}$$

Therefore, it includes W^4 terms coming from the W^2 terms in the field. Now our ansatz consists of identifying some of these W^4 terms, of coefficient $-\frac{1}{4}\, g^2\, (c^\alpha_{\beta\gamma})^2$, with the Higgs boson ϕ^4 term of coefficient $-\frac{1}{2}\, \lambda^2$ (where ϕ is the initial scalar doublet).

Namely, let us first separate the six components of the total field in two sub-systems, $[W_1, W_2, W_3]$ and $[B_1, B_2, B_3]$. The three W's can be identified with the standard SU(2) field and, say, B_1 with the $\mathrm{U(1)_Y}$ field. Two vectorial fields remain, B_2^μ and B_3^μ. They will contribute in a non-vanishing way to the quartic term in the Lagrangian by their cross product. At the approximation (considered here) where their space components are negligible, we find:

$$-B_{2\mu}\, B_{3\nu}\, B_3^\mu\, B_2^\nu = -[B_2^0\, B_3^0]^2. \tag{115}$$

Finally, we make the identification of these time components with the residual Higgs boson, $B_2^0 = B_3^0 = \sigma$. This allows a determination of the constant λ according to the relation:

$$\lambda^2 = \frac{g^2\, c^2}{2}, \tag{116}$$

where the squared Lie coefficient c^2 is equal to 1 in the case of an SO(4) group. Provided the global charge is identical to the SU(2) charge, and since the W mass is given by $m_W = gv/\sqrt{2}$, one finally obtains a Higgs boson mass :

$$m_H = \sqrt{2}\, m_W. \tag{117}$$

Using the recently precisely determined W boson mass, $m_W = 80.42 \pm 0.04\,\mathrm{GeV}$ [27], we have obtained a theoretical prediction [24, 25]:

$$m_H = \sqrt{2}\, m_W = 113.73 \pm 0.06 \text{ GeV}, \tag{118}$$

which is in agreement with current constraints and with a possible recent detection at CERN with a mass of ≈ 114 GeV [27].

However one should keep in mind that this result is obtained in the framework of an approximate theory. The SO(4) group can be only an incomplete group, as well from the view point of the scale-relativity approach (other variables and other scale transformations should be taken into account) as in the standard quantum field theory approach (in which the unifying gauge group should also include the SU(3) of chromodynamics). Although the self-consistency of this model then remains to be established, we hope that at least some of its ingredients could reveal to be useful in more complete attempts.

7 Conclusion and future prospect

In this contribution, we have attempted to reach an understanding from first principles, in terms of a geometric space-time description, of the nature of gauge transformations. Let us indeed recall the fundamental difference between the situation of transformations in the standard gauge theories and transformations whose geometric meaning is known such as , e.g., rotations in space or Lorentz transformations.

We know from the very beginning what Lorentz transformations are, namely, space-time rotations of the coordinates, i.e., in the case of an infinitesimal transformation, (i) $dx'^{\alpha} = (1 + \omega^{\alpha}_{\beta})dx^{\beta}$. Then, once this basic definition is given, one can consider the effect of these transformations on various physical quantities ψ. This involves the consideration of representations of the Lorentz group adapted to the nature of the physical object under consideration, i.e., (ii) $\psi' = (1 + \frac{1}{2}\omega^{\alpha\beta}\sigma_{\alpha\beta})\psi$ (see e.g. [28]).

In the case of the standard theory of gauge transformations, there was up to now no equivalent of the basic defining transformation (i), and the gauge group was directly defined through its action on the various physical objects (ii).

It is precisely an equivalent of the defining transformation (i) that we propose in the scale-relativity framework: namely, we have given a geometric meaning to the gauge transformations. We now interpret them as scale transformations in the scale space. These transformations apply to the fractal structures which characterize the geodesics of a fractal space-time (identified with a particle) at scales smaller than its Compton length: this last point allows one to solve the problem encountered by the Weyl theory.

Then the gauge fields are understood as the manifestation of a general-scale-relativistic effect, i.e. as the geometric effects of dilations and contractions of the internal fractal structures in scale space that are induced by the displacements in space-time: in other words, they correspond to terms of coupling between motion and scale. Finally the charges are identified with the conservative quantities that find their origin in the symmetries of the new scale variables.

On the basis of the first stones recalled here, a huge work of construction remains to be done, which will be described in future publications, namely: (i) the second quantization of fields in the scale-relativity framework; (ii) the identification of the full group of transformation in the non-Abelian case and its decomposition in sub-groups, and the associated identification of the various charges with their geometric meaning; (iii) the possible generalization with these definitions of the mass-coupling relations and their application to prospective theoretical predictions of particle masses; (iv) the development of the new approach to the Higgs boson field and to the symmetry breaking mechanism of the electroweak theory, of which only hints have been given here; (iv) the connection of the scale-relativity description with the renormalization group approach, in particular with regard to the variation of the running couplings and particle masses in function of the scale.

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