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F. Combes, D. Barret, T. Contini, F. Meynadier and L. Pagani (eds)

ENERGY STRUCTURE AT 3.2×10^{20} EV IN SPECIAL SCALE-RELATIVITY

Nottale, L.[1]

Abstract. In scale relativity, one considers a space-time geometry that is continuous but non-differentiable, which implies its fractality (i.e. a structuring of the scale-space). One can show that the standard laws of dilation correspond to a "Galilean" version of the theory. Their Lorentzian generalization involves the appearance of two length-scales, invariant under dilations and unreachable, one toward the small scales that can be identified with the Planck length-scale and the other toward the large scales that can be identified with the scale $\mathbb{L}$ of the cosmological constant ($\Lambda = 1/\mathbb{L}^2$). One of the consequences of these new laws of dilation is the existence of a very high energy structure at $E = (3.2 \pm 0.2) \times 10^{20}$ eV, generated when the SU(2) running coupling reaches the critical value $1/4\pi^2$. This predicted energy is precisely the maximal energy observed for ultra-high energy cosmic rays by the Fly's Eye detector, at $E = (3.2 \pm 0.9) \times 10^{20}$ eV.

1 Introduction

Ultra high energy cosmic rays observed beyond 10^{19} eV are still of unknown origin. The highest energy shower has been observed in 1991 by the Fly's Eye detector at $(3.2 \pm 0.9) \times 10^{20}$ eV (Bird et al., 1994, 1995). The second highest energy event is at 2.1×10^{20} eV. Such detections, and more generally the highest energy spectrum of cosmic rays, pose an important problem, because one expects a Greisen-Zatsepin-Kuzmin (GZK) cutoff of the energy spectrum at $\approx 4 \times 10^{19}$ eV for travel distances larger than about 30 Mpc, due to pion photoproduction energy losses. Such a constraint strongly limits the possible astrophysical sources for such events, provided they are produced by accelerated known particles.

In order to circumvent this problem, it has been proposed that the highest energy cosmic rays originate in the decay of topological spacetime defects such as cosmic strings or vortons (Bonazzola & Peter, 1997). Such theories would predict a continuing cosmic ray flux all the way up the grand-unification mass scale (10^{23} eV in the minimal standard model, 10^{28} eV if the unification is at the Planck energy).

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2 Ultra-high energy structure in special scale-relativity

A similar proposal can be made in the scale-relativity framework (Nottale, 2003), but with another mass scale for the primary particle. Recall that the special scale-relativity theory (Nottale, 1992) introduces a generalized law of scale transformations of a log-Lorentz form, in which the Planck length-scale becomes a limiting, impassable, minimal scale in nature, invariant under dilations and contractions, playing the role previously devoted to the zero point. Moreover, in this new framework we expect the occurrence of new kinds of spacetime structures, linked in particular to mass-charge relation and to the critical value $4\pi^2$ of inverse couplings (Nottale 1996). One of these structures is given by the equation (see Fig. 1):

$$\alpha_2^{-1}(E) = 4\pi^2, \tag{2.1}$$

where α_2 is the SU(2) weak coupling and E is the energy-scale to be solved for.

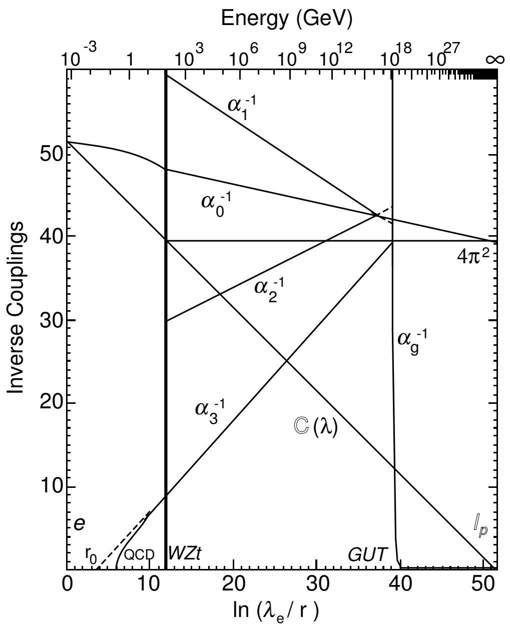

Fig. 1. Variation with scale of the inverse couplings of the fundamental interactions U(1), SU(2) and SU(3) in the scale-relativistic minimal standard model. This scale-relativistic diagram (mass-scale versus inverse coupling constants) shows well-defined structures and symmetries which are predicted by the theory (Nottale, 1996)).

The scale dependence of the α_2 running coupling is given by the solution to its renormalization group equation, that reads to first order (see e.g. Nottale 1993, and references therein):

$$\alpha_2^{-1}(r) = \alpha_2^{-1}(\lambda_Z) + \left(\frac{5}{3\pi} - \frac{N_H}{12\pi}\right) \ln\left(\frac{\lambda_Z}{r}\right), \tag{2.2}$$

where λ_Z is the Compton length of the Z boson, N_H is the number of Higgs doublets, and r a running length-scale. As demonstrated in (Nottale, 1992, 1993), this solution remains correct in the special scale-relativity framework, provided it is written in terms of length-scale. Conversely, while one can replace $\ln(\lambda_Z/r)$ by $\ln(m/m_Z)$ in the standard model, this is no longer the case in special scale-relativity, since a log-Lorentz factor is now involved in the mass-scale / length-scale transformation, namely,

$$\ln\left(\frac{\lambda_Z}{r}\right) = \frac{\ln(m/m_Z)}{\sqrt{1 + \ln^2(m/m_Z)/\ln^2(m_{\mathbb{P}}/m_Z)}} \tag{2.3}$$

Now solving Eq. 2.1 for $E = mc^2$ with the precise value of $\alpha_2^{-1}(m_Z) = 29.802 \pm 0.027$ (PDG, 2000) yields:

$$E = (3.20 \pm 0.26) \times 10^{20}\text{eV}. \tag{2.4}$$

Including second order corrections in the renormalization group equation and accounting for the scale-relativistic correction on the fundamental constant $\mathbb{C}_Z = \ln(\lambda_Z/l_P) = 39.756$ (which differs by 1% from $\ln(m_P/m_Z) = 39.436$ (Nottale, 1993), one obtains an equivalent result,

$$E = (3.27 \pm 0.26) \times 10^{20}\text{eV}, \tag{2.5}$$

which agrees very closely with the maximal energy of cosmic rays observed at $(3.2 \pm 0.9) \times 10^{20}$ eV (see Fig. 2). Due to the large value of this energy, the agreement would remain remarkable even if the experimental error revealed to be underestimated.

Such a result, provided it is confirmed by future observations, e.g. at the Pierre Auger observatory, allows one to put to the test the number of Higgs doublets and the scale-relativistic log-Lorentz factor. Indeed, the predicted energy becomes 8.0×10^{19} for 0 Higgs doublet (excluded) and 1.7×10^{21} for 2 Higgs doublets (which will be excluded if no showers at energies larger than 3.2×10^{20} eV are observed). This is a confirmation of our result concerning the fine structure constant (Nottale, 1996):

$$\alpha^{-1} = (137.04 \pm 0.03) + 2.11(N_H - 1), \tag{2.6}$$

that already strongly excludes $N_H \neq 1$.

This effect could also be used to discriminate between the Galilean scale-relativistic (i.e. standard) and Lorentzian scale-relativistic theory. Indeed, in the absence of a log-Lorentz factor, one obtains for one Higgs doublet $E = (2.0 \pm 0.12) \times 10^{19}$ eV, which is smaller than the Fly's Eye energy by a factor 16 and lies below the GZK cutoff.

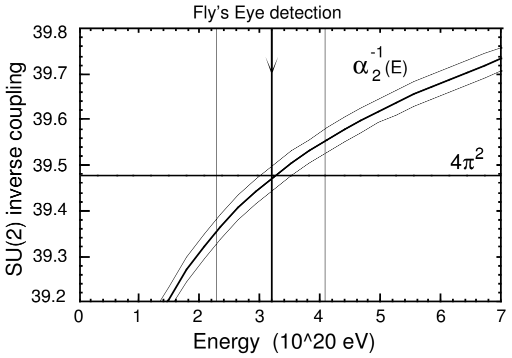

Fig. 2. Comparison of the theoretical prediction with the Fly's Eye detection (see text).

With additional Higgs doublets, one obtains $(1.9 \pm 0.1) \times 10^{20}$ eV for 3 doublets and 7.2×10^{20} eV for 4 doublets. The first value is already too low and the second will be excluded if the present limiting energy is not exceeded by future detections. This is therefore a new test of the log-Lorentz factor, that is added to other previous tests (Nottale, 1992, 1993).

3 Conclusion

Such a proposal, namely the decay of a new particle of mass 3.2×10^{20} eV/c^2, is falsifiable, since in this case one would not expect to find cosmic rays of energy far larger than the Fle's Eye value. Moreover, if it were confirmed, the distance limits set on the source no longer apply. This reopens the possibility of a galactic source (as supported by the arrival direction that lies close to the galactic plane, at $b = 9.6$ deg), or of a very distant extragalactic source: hence 3C147, a QSO of redshift 0.545, lies within the 1σ error box (Elbert & Sommers 1995).

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GENERALIZED MACROSCOPIC SCHRÖDINGER EQUATION IN SCALE RELATIVITY

Célérier, M.N.[1] and Nottale, L.[1]

Abstract. The scale transformation laws produce, on the motion equations of gravitating bodies and under some peculiar assumptions, effects which are anologous to those of a "macroscopic quantum mechanics". When we consider time and space scales such that the description of the trajectories of these bodies (planetesimals in the case of planetary system formation, interstellar gas and dust in the case of star formation, etc...) is in the shape of non-differentiable curves, we obtain fractal curves of fractal dimension 2. Continuity and non-differentiability yield a fractal space and a symmetry breaking of the differential time element which gives a doubling of the velocity fields. The application of a geodesics principle leads to motion equations of Schrödinger-type. When we add an outside gravitational field, we obtain a Schrödinger-Poisson system. We give here the derivation of the Schrödinger equation for chaotic systems, i.e., with time scales much longer than their Lyapounov chaos-time.

1 Introduction: the foundations of scale relativity

The scale relativity theory is a geometric representation of nature (as General Relativity is a geometric representation of gravitation) based on a continuity hypothesis and constrained by the relativity principle. It includes in its description non-differentiable manifolds, thus the fractal character of space-time in the general meaning: fractal $\equiv \mathcal{L}_{D_T}(\epsilon) \underset{\epsilon \to 0}{\longrightarrow} \infty$ (for the demonstration see (Nottale, 1993)).

This approach involves a scale dependence of the reference frames. We therefore add to the standard variables (position, orientation and motion), which characterize the reference frame, other variables characterizing its scale state. The use of differential equations is made possible thanks to the representation of physical quantities, usually mere functions of the space-time coordinates $f(x)$, by fractal functions $f[x(\epsilon), \epsilon]$, explicitly depending on the scale variables, generically noted ϵ.

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Generalizing the definition of fractal functions to fractal space(-time)s (fractal space(-time) $\equiv$ equivalence class of a family of Riemannian space(-time)s), we obtain scale dependent geodesics equations and, therefore, an infinite family of geodesics.

2 Dynamics in scale relativity

In the scale relativity theory, the Schrödinger equation is derived from three fundamental conditions which are consequences of the non-differentiability:

(1) The fractality of space, which implies that the number of geodesics is infinite. We are therefore led to use a fluid-like description where the velocity $v(t)$ is replaced, as a first step, by a velocity field $v[x(t), t]$.

(2) The fractal geometry of each geodesic, which implies that the velocity field is actually a fractal function, $V[x(t, dt), t, dt]$, explicitly depending on a scale variable, identified, in the present case, to the differential element dt. One can show (Nottale, 1993) that it can be decomposed in terms of the sum of a classical (differentiable) velocity field and of a divergent fluctuation field,

$$V[x(t, dt), t, dt] = v[x(t), t] + w[x(t, dt), t, dt] = v \left[1 + \zeta \left(\frac{\tau}{dt}\right)^{1-1/D_F}\right], \qquad (2.1)$$

where D_F is the fractal dimension of the geodesics. The w function is a fractal fluctuation which is described in terms of a stochastic variable such that (for the critical case $D_F = 2$)

$$< w_i >= 0 \qquad\qquad < w_i w_j >= \delta_{ij} \left(\frac{2\mathcal{D}}{dt}\right). \qquad (2.2)$$

(3) The non-differentiability of space, which breaks the local reflection invariance of the time differential element dt. As a result, two fractal velocity fields V_+ and $V_-$are defined, which are fractal functions of the scale variable dt. Each velocity field split, as in Eq.(2.1), into

$$V_+ = v_+[x(t), t] + w_+[x(t, dt), t, dt], \quad V_- = v_-[x(t), t] + w_-[x(t, dt), t, dt]. \quad (2.3)$$

2.1 *Covariant derivative operator*

Even after the transition to the "classical" domain is completed, there is no reason for the two velocities v_+ and v_- to be equal. The natural choice for a mathematical representation of this twin-process is the use of complex numbers (Célérier and Nottale, 2004). The elementary displacement for each of the two processes, $dX_\pm$, can thus be written as the sum of a scale independent "classical" term and a fluctuation around this term,

$$dX_+(t) = v_+ \; dt + d\xi_+(t), \quad dX_-(t) = v_- \; dt + d\xi_-(t). \qquad (2.4)$$

Two “classical” derivative d/dt_+ and $d/dt_-$are defined , which are applied to the position vector x to obtain the two “classical” velocities,

$$\frac{d}{dt_+}x(t) = v_+ \quad \frac{d}{dt_-}x(t) = v_- . \tag{2.5}$$

To recover local reversibility of the time differential element, the two derivatives are combined in terms of a complex derivative operator,

$$\frac{\acute{d}}{dt} = \frac{1}{2}\left(\frac{d}{dt_+} + \frac{d}{dt_-}\right) - \frac{i}{2}\left(\frac{d}{dt_+} - \frac{d}{dt_-}\right), \tag{2.6}$$

which, when it is applied to the position vector, gives a complex velocity,

$$\mathcal{V} = \frac{\acute{d}}{dt}x(t) = V - iU = \frac{v_+ + v_-}{2} - i\,\frac{v_+ - v_-}{2}. \tag{2.7}$$

Now, the total derivative with respect to t of a function $f(x,t)$ contains finite terms up to the highest order. For a fractal dimension $D_F = 2$, it writes

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \nabla f.\frac{dX}{dt} + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}\frac{dX_i dX_j}{dt}. \tag{2.8}$$

The “classical” scale independent part of the term $dX_i dX_j/dt$ is finite and equal to $< d\xi_i\ d\xi_j > /dt = \pm 2\,\mathcal{D}\,\delta_{ij}$. The last term of the scale independent part of this equation is therefore a Laplacian, and the final expression for the complex time derivative operator is derived (Nottale, 1993)

$$\frac{\acute{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta \ . \tag{2.9}$$

2.2 *Improving the covariant tool of scale relativity*

This operator is a linear combination of first order and second order derivatives, so that its Leibniz rule is also a linear combination of the first order and second order Leibniz rules. Now the covariant character of this tool can be improved by introducing a ‘symmetric product’ (Pissondes, 1999) in terms of which the first order form is recovered. Another solution consists of defining a complex velocity operator, whose non-relativistic version is (Nottale, 2004)

$$\widehat{\mathcal{V}} = \mathcal{V} - i\,\mathcal{D}\,\nabla \ . \tag{2.10}$$

The covariant derivative is now written as an expression that keeps the standard (first order) form of the decomposition of a total derivative into partial derivatives, namely

$$\frac{\acute{d}}{dt} = \frac{\partial}{\partial t} + \widehat{\mathcal{V}}.\nabla \ . \tag{2.11}$$

More generally, one defines the operator:

$$\frac{\widehat{\acute{d} f}}{dt} = \frac{\acute{d} f}{dt} - i\mathcal{D} \, \nabla f . \nabla \ . \tag{2.12}$$

The covariant derivative of a product now writes $\acute{d}\,(fg)/dt = g\, \widehat{\acute{d} f/dt} + f\, \widehat{\acute{d} g/dt}$ i.e., one recovers the form of the first order Leibniz rule for products. Thanks to this formal tool, the standard form of the equations is preserved, i.e., a full covariance under the generalized transformations considered here is ensured.

2.3 Newton-Schrödinger equation

Standard classical mechanics can now be generalized using this covariant tool. The application of a Lagrangian formalism yields the scale relativistic Euler-Lagrange equations (Nottale, 1993), i.e.,

(1) For the case of inertial motion, a geodesics equation: $\acute{d}\,\mathcal{V}/dt = 0$.

(2) For the case when the external structuring field is a scalar potential Φ, a Newton-type equation of dynamics: $m\acute{d}\,\mathcal{V}/dt = -\nabla\Phi$.

A complex wave function is introduced, which is another expression for the complex action $\mathcal{S}$: $\psi = e^{i\mathcal{S}/\mathcal{S}_0}$. We substitute it into the Euler-Lagrange equation, as well as the complex velocity which is the gradient of the complex action: $\mathcal{V} = \nabla\mathcal{S}/m$. The choice $\mathcal{S}_0 = 2m\mathcal{D}$ finally allows to write this equation as a gradient, which, after integration, yields the Newton-Schrödinger equation (Nottale, 1993),

$$\mathcal{D}^2 \Delta\psi + i\mathcal{D}\frac{\partial}{\partial t}\psi = \frac{\Phi}{2m}\psi \ . \tag{2.13}$$

3 Conclusion

We have recalled how, under three general conditions involving non-differentiability and fractality, the fundamental equation of dynamics can be transformed to take the form of a generalized Schrödinger equation. Such an equation is naturally structuring, since, once the potential, the symmetry and limiting conditions are specified, its solutions yield probability densities that describe the tendency for the system to make structures (Nottale, 1997). Various applications of this approach are given in other contributions to the present issue.

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STRUCTURE FORMATION BY THE HARTREE EQUATION

Lehner, T.[1], Nottale, L.[1], di Menza, L.[2], da Rocha, D.[1] and Ceccolini, D.[1]

Abstract. The Scale Relativity theory predicts that the formation of structures in gravitational interaction can be described by a Hartree equation or equivalently by a set of coupled Schrödinger and Poisson equations. This system is similar to the one used in quantum gravity (with matter only being quantized, not the field), but with the substitution of $(\hbar/m)$ by a parameter depending of the system under study. As applications we show the formation of structures in a medium with a homogenous density and the formation of a disk around a central star. We describe also the phase of nonlinear dynamical evolution which is thought to converge asymptotically in time towards the equilibrium linear solutions.

1 Introduction

In scale relativity (see e.g., Célérier & Nottale, 2004 and this issue), in order to describe gravitational structure formation one is led to write an Hartree-like equation, similar to the one encountered in quantum gravity systems as :

$$i\mathcal{D}\frac{\partial\psi}{\partial t} + \mathcal{D}^2\Delta\psi - \frac{(\phi + \phi_{ext})}{2m}\psi = 0, \tag{1.1}$$

$$\Delta\phi = 4\pi Gm\rho = 4\pi Gm\,|\psi|^2\,, \tag{1.2}$$

where $\mathcal{D}$ depends of the size of the considered system (within a multifractal description). In Eqs (1.1-1.2) $\psi(\mathrm{x}, t)$ is the wave function of an auto-gravitating system of mass m with a gravitational potential energy ϕ in presence of a possible external potential energy ϕ_{ext}. Matter is quantized by the choice $\rho = |\psi|^2$. In such a description, structure formation is expected to be the signature of quantum-like effects at the macroscopic level (which take their origin in the fractality of space, see Nottale, 1993). Two main situations will be considered :

a) $\phi_{ext} = -GmM/r$ (Kepler potential) with a main star of mass M, suitable to describe accretion disk formation;

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b) $\phi_{ext} = Ar^2$ (harmonic potential) suitable to describe large scale structures from a cosmological background density ρ_b, with $A = 2\pi Gm\rho_b/3$.

Two-dimensional spatial versions of a) and b) have been also studied, with in case a) ϕ_{ext} behaving as $\ln r$.

In a previous study (da Rocha & Nottale, 2003) linear eigensolutions of the system (i.e., without equation (1.2)) for the wave function in presence of the external potential have been only given. These solutions predict structures which are strikingly similar to observations (da Rocha & Nottale, 2003; Galopeau et al., this issue). Here we investigate numerically the nonlinear stage of these equations.

2 Short description of the numerical code and some mathematical properties

We have designed a code to solve the coupled Schrödinger and Poisson (or Hartree) equations. This code uses both a spatial (1D, 2D or 3D) and temporal finite differences written in Cartesian units (a radial 1D code is also available) and it computes the time evolution of the solution for a given initial condition (Cauchy problem), $\psi(\mathrm{x}, t) = 0$. We use a Crank-Nicolson symmetrical implicit scheme which is unconditionally stable. At each time step, a nonlinear discrete algebraic system is solved by a fixed point method. Tests are showing a good conservation by the discretization procedure of the mass, the energy and the angular momentum.

Some mathematical results are known about the Hartree equations (in particular without external potential) and we give only some of them here :

−Contrary to the nonlinear Schrödinger equation (which is close to the Hartree equation by taking a singular limit) with a focusing nonlinearity the solutions are global in time for regular enough initial conditions.

−Theorems also prove the existence of bound states for this equation since the energy of the system is bounded by below; these nonlinear bound states can be of the soliton type and are the equivalent of the eigen-energy levels for the linear Schrödinger equation in a confining external potential.

−The energy ε and the mass M are fundamental invariants of the system and we have $\mathrm{d}\varepsilon/dt = 0$ and $dM/dt = 0$, where

$$M = \int |\psi|^2 d^n x, \varepsilon = \int (\nabla\psi.\nabla\psi^* + \frac{1}{2}\Phi|\psi|^2) d^n x. \qquad (2.1)$$

−The T kinetic energy and the V_p potential energy are defined by

$$T = \int \nabla|\psi|^2 d^n x, \;\; V_p = \int \varphi|\psi|^2 d^n x, \;\; \varepsilon = T + \frac{1}{2}V_p. \qquad (2.2)$$

− In the stationnary case of energy eigenvalue E we have a nonlinear virial theorem as $T = -\frac{1}{3}E,\ V_p = \frac{4}{3}E,\ \varepsilon = \frac{1}{3}E$. It is the equivalent of the familiar linear virial equipartition theorem for which $< T >= \frac{n}{2} < V >$ if $V \sim r^n$.

In particular all the energy eigenvalues are negative for the bound states.

− A property of dispersion of the wave function has been shown as follows: for the momentum $Q = \int |x|^2|\psi|^2 d^n\mathrm{x}$, if the initial data are smooth enough for Q to be defined, one gets the relation $Q'' = 8\varepsilon - 2V_p$. If ϕ is everywhere negative then V_p is negative also. If also ε is positive, then Q will grow quadratically with time: a solution with a positive energy will be hence dispersive contrary to the bound states of negative energy.

3 Numerical procedures and results

Solutions without external potential (purely autogravitating sytems). −*Bound states*: We have looked first for the bound localized radial (stationnary states) of the Hartree equation by means of the shooting method. Solutions are searched in the form $\psi(x,t) = e^{\imath\omega t}u(x)$ with $\psi(x,0) = u_0(x)$, with normalization of the equations such that $\mathcal{D} = 1$ and $4\pi G = 1$ in Eqs (1.1-1.2) with $\phi_{ext} = 0$.

We have to solve the equations (for ω given):

$$-\omega u + \Delta u - \phi u = 0, \quad \Delta\phi = uu^*, \quad \lim_{\|x\|\to+\infty} u(x) = 0. \tag{3.1}$$

This step is achieved using a simpler code involving a radial ordinary differential equation. For example, we have found the solutions in 3D for the fundamental radial ground state and for the excited state. Results have been also obtained for 1D and 2D spatial dimensions.
−*Time evolving solutions*: The stationnary above solution for ψ is injected in the time dependent code and we have checked that its remains stable with time; results were obtained in 2D and 3D spatial dimensions for various initial conditions.

The Keplerian case. We first look for the stationnary states in 3D of the normalized equations (1.1-1.2) with a ϕ_{ext} scaling in w_0/r where $\mathcal{D} = GM/2w_0$, w_0 being a velocity, typically $w_0 = 144$ km/s for the inner solar system and the presently discovered extrasolar planetary systems (see Galopeau et al., this issue).

We have found bound states in this case and we have implemented them as initial conditions in the time evolving code showing the stability in time of the solution, see Fig. 1. Note that the radial width of the localized state becomes only slightly shrinked and yields a typical size structure.

-2D case: various sub-cases have been studied with an initial mass ring and a central star for a $1/r$ potential but also for a true 2D $\ln r$ Coulomb potential. Both cases (not presented here but in a future publication) show a tendency to accretion and disk formation.

-3D case : we have tried various conditions like an initial 3D torus or with an initial crown of matter. In both cases outer matter is attracted towards the center forming structures.

The harmonic potential case. In this case due to the growth of the external potential in r^2 at infinity we have not yet found possible stationnary solutions. But we have implemented the time dependent equations with a Gaussian radial initial solution since the solutions of the corresponding linear Schrödinger equation are

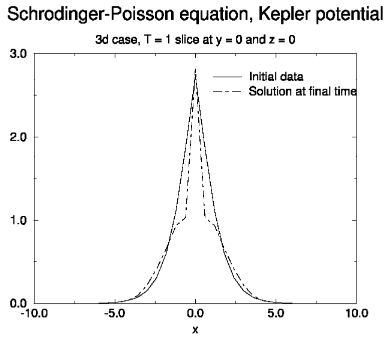

Fig. 1. $\psi(x,t)$ for autogravity and $1/r$ Kepler potential of equal strength.

the Hermitte polynomials. Here a charateristic size is given by $a = (2\mathcal{D}/\omega)^{1/2}$ with $\omega = (3/2\pi G\rho_b)^{1/2}$, while the coefficient $\mathcal{D}$ can be deduced from the observations.

The figure 2, for example, shows a simulation in 3D with a conservation of the initial central Gaussian shape versus time.

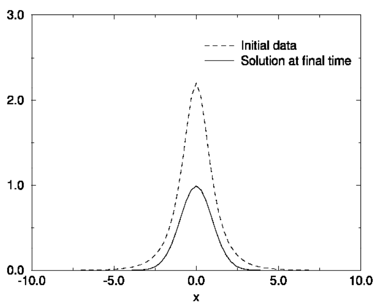

Fig. 2. $\psi(x,t)$ for autogravity and r^2 harmonic potential of equal strength.

4 Conclusions

We have shown that the Hartree equation can lead to structure formation in its nonlinear stage. The bound states are found to be stable in time. This confirms previous studies using the linear Schrödinger equation on the possible time asymptotic states and their relevant morphology in connection with the observations.

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SCALE RELATIVITY AND DARK POTENTIAL

Nottale, L.[1]

Abstract. In the framework of the theory of scale relativity, one considers a geometry that is not only curved, but also non-differentiable and therefore fractal, i.e. structured along scales. The equation of dynamics (i.e. of geodesics) in such a space can be integrated in terms of a generalized Schrödinger equation. Then one can show that, in analogy with the curvature manifesting itself as the Newton potential (to first approximation), the fractality leads to the appearance of a new potential energy. This result allows one to suggest a new alternative solution, of geometrical nature, to the problem of the effects that are usually attributed to unseen, non baryonic dark matter.

1 Introduction

Recall that up to now two hypotheses have been formulated in order to account for the numerous effects which are unexplainable by the sole action of the gravitational force of visible matter (flat rotation curves of spiral galaxies, large velocity dispersion of clusters of galaxies, cosmological energy balance, gravitational lensing, formation and evolution of structures, ...):

(i) The existence of a very large amount of unseen matter in the Universe: however, despite intense and continuous efforts, it has escaped detection.

(ii) A modification of Newton's law of force: however, such an ad hoc hypothesis seems impossible to reconcile with the geometric origin of gravitation in general relativity, which lets no latitude for modification.

An alternative proposal (Nottale, 2001, 2003; Da Rocha & Nottale, 2003) is recalled in this contribution. In the scale-relativity approach, the space-time geometry is not only curved, but also fractal (beyond some relative transition scale). This fractality manifests itself in terms of an additional potential energy. We can therefore consider the new possibility that this "dark potential" be the cause of the non-Newtonian effects. In such a proposal, there would be no need for additional non-baryonic matter, and Newton's potential would be unchanged since it remains linked to the mere curvature.

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2 New scalar field manifesting the fractality of space-time

Recall that, starting from three minimal conditions arising from non-differentiability, namely, {(i) infinity of geodesics; (ii) decomposition of each elementary displacement in terms of the sum of a classical variable and a fractal variable of fractal dimension 2; (iii) two-valuedness of the velocity vector due to irreversibility in the reflexion ($dt \leftrightarrow -dt$)}, the geodesics equation in a curved and fractal space(-time),

$$m \frac{\hat{d} \mathcal{V}}{dt} + \nabla \phi = 0, \tag{2.1}$$

can be integrated in the form of a generalized Schrödinger equation (see e.g. Célérier & Nottale, this issue) that writes at the Newtonian limit:

$$\mathcal{D}^2 \Delta \psi + i \mathcal{D} \frac{\partial}{\partial t} \psi = \frac{\phi}{2m} \psi, \tag{2.2}$$

where $\mathcal{D}$ characterizes the fractal fluctuation, and ϕ is the Newtonian potential energy, which is a solution of the Poisson equation:

$$\Delta \phi = 4 \pi G m \rho. \tag{2.3}$$

From our description of the motion in terms of an infinite family of geodesics, the meaning of $P = \psi \psi^\dagger$ is imposed as giving the probability density of the particle positions, in agreement with Born's postulate (Célérier & Nottale, 2004). Indeed, separating the real and imaginary parts of the Schrödinger equation and writing it in terms of P and of the classical velocity V (which is the real part of the complex velocity $\mathcal{V}$), we obtain respectively a generalized Euler-Newton equation and a continuity equation (Nottale et al., 2000):

$$\left(\frac{\partial}{\partial t} + V \cdot \nabla \right) V = -\nabla \left(\frac{\phi + Q}{m} \right), \tag{2.4}$$

$$\frac{\partial P}{\partial t} + \mathrm{div}(PV) = 0. \tag{2.5}$$

This system of equations is equivalent to the classical one used in the standard approach of gravitational structure formation, except for the appearance of an additional potential energy Q which reads:

$$Q = -2m \mathcal{D}^2 \frac{\Delta \sqrt{P}}{\sqrt{P}}. \tag{2.6}$$

This potential energy is a manifestation of the fractality of space, in the same way as the Newtonian potential is a manifestation of space-time curvature. Therefore this system of equations is similar to that used under the dark matter hypothesis, since in that case one also adds to the potential ϕ due to visible matter (and to small amounts of baryonic dark matter), an additional potential ϕ_{DM} which is linked through a Poisson equation to the postulated non-baryonic dark matter density. However, there is a fundamental difference (in addition to the fact that here the new potential energy Q is not postulated, but derived from first principles), since now the link of Q to the density is not Poissonian.

3 Energy balance

We are therefore led to investigate the possibility that its existence may contribute explaining the various dynamical effects presently attributed to unseen dark matter. Indeed, let us come back to the Schrödinger form of these equations. Two extreme situations (and any intermediate case between them) can be considered:

(i) The particles fill the probability density distribution, so that the density of matter is proportional to the density of probability, $\rho \propto P$. In this case the system of equations is a coupled Schrödinger-Poisson (Hartree) system. The general existence of steady-state solutions with conserved total energy has been demonstrated for this system (see refs. in Nottale, 2003). This case corresponds to a self-gravitating body, such as a cluster of galaxies.

(ii) There are only very few test-particles, so that from the view-point of matter density, we deal with the vacuum. This case corresponds e.g. to the outer regions of spiral galaxies (in the absence of large amounts of dark matter as assumed here). Therefore ϕ is a solution of $\Delta\phi = 0$, i.e. $\phi = -Gm\,\Sigma_i(M_i/r_i)$. The Schrödinger equation with such a potential does have general stationary solutions.

Therefore in both cases, we can write a time-independent Schrödinger equation:

$$2m\mathcal{D}^2\Delta\psi + (E-\phi)\,\psi = 0, \tag{3.1}$$

where ϕ is the steady-state solution for the potential. In the gravitational macroscopic case considered here, this equation is subjected to the principle of equivalence (contrarily to the standard microscopic quantum mechanics, where $\mathcal{D} = \hbar/2m$), and therefore it does not depend on the inertial mass m of the bodies whose distribution is described. For steady-state solutions, one finds the general energy balance relation:

$$E = \phi + Q + \frac{1}{2}mV^2(x,y,z), \tag{3.2}$$

which is therefore increased with respect to the standard description by the contribution of the new potential energy.

4 Applications

Let us briefly apply this method to two typical cases of astrophysical situations where anomalous dynamical effects (usually accounted for in terms of dark matter) have been identified.

4.1 Flat rotation curves of spiral galaxies

Let us consider a very simplified argument. Under the hyspothesis that there is no matter beyond the visible radius r_0, the virial theorem yields $\phi = -2E_c$, so that $v = \sqrt{GM/r}$, where M is the galaxy mass, while a constant velocity $v = v_0$ is observed. Now, in a Kepler potential the additional potential energy (that comes from the fractality of trajectories, see Fig. 1) reads $Q = -(GMm/2r_B)(1-2r_B/r)$, so that one obtains a constant velocity beyond r_0, provided $r_0 = 2r_B$.

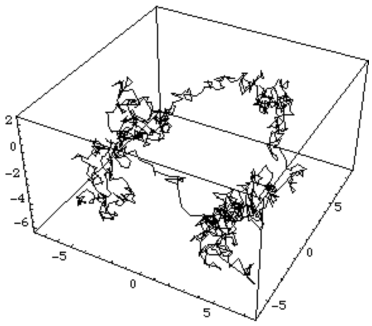

Fig. 1. Example of numerical simulation of trajectory in a Kepler central potential, based on the basic scale-relativity equations.

4.2 Velocity dispersion of clusters of galaxies

The description of clusters of galaxies comes under the Hartree system, which should be solved numerically. However, one can use the fact that the solution will be subjected to a generalized Heisenberg relation that reads $\sigma_x \times \sigma_v \approx 2\mathcal{D} = GM/2w_0$, where w_0 is a universal constant which characterizes the fractal fluctuation (see e.g. Da Rocha & Nottale 2003). Its value has been found to be $w_0 = 144$ km/s in several systems at several various scales. In this case a typical velocity dispersion for large clusters of ≈ 800 km/s can be obtained for a small cluster mass $M \approx 5 \times 10^{13}$ $M_\odot$ that does not include excedentary non-baryonic dark matter, since we get the numerical relation: (1 Mpc) $\times$ (800 km/s) = G $\times(5 \times 10^{13}$ $M_\odot)/2\times$ (144 km/s).

5 Conclusion and prospect

This new road of research about the 'dark matter problem' has given encouraging preliminary results, so that we intend to develop it along the following lines: (i) research of analytical and numerical solutions of the Schrödinger-Poisson system (see Lehner et al, this issue); (ii) meaning of the new potential Q and of the fluctuation coefficient $\mathcal{D}$; (iii) application to gravitational lensing; (iv) application to the cosmological energy balance.

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DISTRIBUTION OF ORBITAL ELEMENTS OF PLANETS AND EXOPLANETS IN SCALE RELATIVITY

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Abstract. In the framework of scale relativity, we describe the motion of planetesimals in the protoplanetary nebula in terms of a fractal and irreversible process. As a consequence the equation of dynamics can be transformed to take a Schrödinger-like form. Its solutions yield a planetesimal distribution showing peaks of probability for particular values of conservative quantities such as the energy and the Runge-Lenz vector. After accretion, this results in expected probability peaks of the semi-major axis distribution at $a_n = (GM/w^2)n^2$, and of the eccentricity distribution at $e = k/n$, where k and n are integer numbers, M is the star mass and w is a constant having the dimension of a velocity. The current observational data in our solar system and extrasolar planetary systems support these predictions in a statistically significant way: we show that these systems are hierarchically organized in terms of a sequence of constants which are multiples and submultiples of $w = 144.7 \pm 0.5$ km/s. New validations of the theoretical predictions are given concerning the very inner solar system including the Sun itself, the distant Kuiper belt and exoplanets recently discovered very close to their star (at about 0.02 AU/$M_\odot$).

1 Distribution of Semi-Major Axes and Eccentricities of Planets

Several features of the newly discovered exoplanets are in good agreement with the predictions derived from the theory, in particular: (i) the accumulation of exoplanets around the same a/M values as the planets of our own solar system (notably around 0.043 UA/$M_\odot$ which corresponds to a Keplerian velocity of $w_0 =$ 144 km/s); (ii) the existence of large eccentricities and probability density peaks for their possible values; (iii) the existence of imbricated levels of organization for planetary systems (in correspondence with the intramercurial, inner and outer systems in our solar system).

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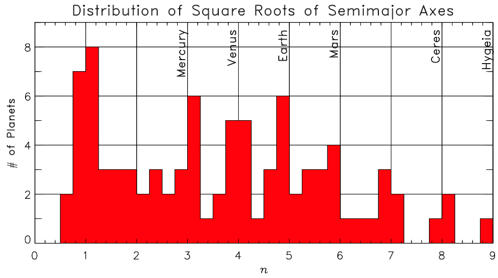

Fig. 1. Histogram of $n = w_0(P/2\pi GM_\star)^{1/3}$ for exoplanets and inner solar system.

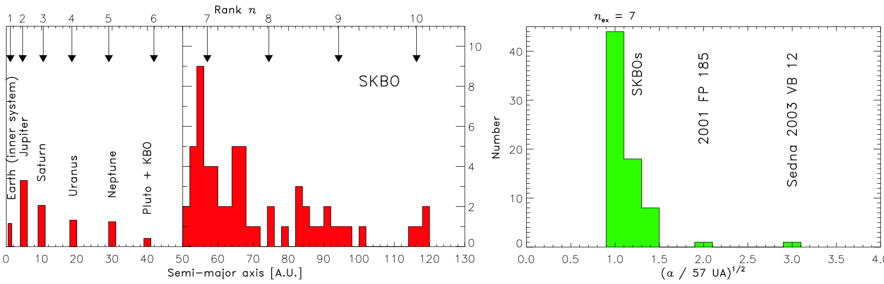

Fig. 2. Left: histogram of the semi-major axes for the outer solar system. Right: histogram of the distribution of $n = w(P/2\pi GM_\star)^{1/3}$ for the very outer solar system (Sedna and scattered Kuiper belt objects) with base $w = 144/(5 \times 7) = 4.11$ km/s.

2 Period of Solar Activity Cycle

We expect the solar surface activity to be subjected to a fundamental period (which is nothing but the macroscopic equivalent of a de Broglie period for the Sun) given by: $\tau = 2\pi m\mathcal{D}/E = 4\pi\mathcal{D}/(v_{\rm rot}^2 + v_{\rm turb}^2) = 2\pi GM_\odot/w_\odot(v_{\rm rot}^2 + v_{\rm turb}^2)$ where the energy E results from the rotational velocity $v_{\rm rot}$ and the turbulent velocity $v_{\rm turb}$ and $\mathcal{D} = GM_\odot/2w_\odot$. The average sideral rotation period for the Sun is 25.38 days, yielding a velocity $v_{\rm rot} = 2.01$ km/s, the turbulent velocity has been found to be $v_{\rm turb} = 1.4 \pm 0.2$ km/s. Taking $w_\odot = 3 \times 144$ km/s (which corresponds to the Keplerian velocity near the Sun radius), we find: $\tau = 10.2 \pm 1.0$ yr whereas the observed period of the solar activity cycle is $\tau_{\rm obs} = 11.0$ yr.

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