Mach's Principle, Dirac's Large Numbers, and the Cosmological Constant Problem.

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Abstract. We assume that the cosmological constant Λ is the sum of a general-relativistic term and of the quantum, scale-varying, gravitational self-energy of virtual pairs. A renormalization group approach is used to describe its scale-dependence. We argue that the large scale value of Λ is reached at the classical electron scale. This reasoning provides us with a large-number relation: $\alpha\,(m_{\mathbb{P}}/m_e) = (\mathbb{L}/l_{\mathbb{P}})^{1/3}$, (where $\mathbb{L} = \Lambda^{-1/2}$, α is the fine structure constant, m_e the electron mass, $m_{\mathbb{P}}$ and $l_{\mathbb{P}}$ the Planck mass and length), which yields a value of Λ in agreement with present observational limits.

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Several unsolved problems plague present cosmology. One of the most serious is the cosmological constant problem: the theoretical expectations of Λ exceed observational limits by some 120 orders of magnitude.[1,2] Indeed it has been realized that the vacuum energy density acts just like a cosmological constant Λ. The trouble comes from the fact that, when it is estimated from quantum field theory, this density is of the order of the Planck density, $\rho_{\mathbb{P}} = c^5/\hbar G^2 \approx 5 \times 10^{93}$ g/cm^3, while the observational bound[2] is currently $\rho_V = \Lambda c^2/G \lesssim 4 \times 10^{-29}$ g/cm^3. Such a combination of microphysical and cosmological concepts, along with the ratio $\approx(10^{40})^3$ of these two quantities suggests to us that this question may be related to two other fundamental proposals, namely Mach's principle and Dirac's large-number coincidences.

There are actually 3 “levels” of what is usually called “Mach's principle”, each of which corresponds to increasingly profound relations between the local and the global physics:

(i) The first consists in requiring that the *inertial frames* are determined by the distant masses: Einstein's theory of general relativity satisfactorily implements this principle.[3]

(ii) The second (‘Mach-Dirac's principle’) contemplates the possibility that the masses of elementary particles are related to structures of the universe as a whole.[7] This hope is founded on the Eddington-Dirac “large-number” coincidences:

$$\text{(a)}\ \frac{m_{\mathbb{P}}}{m} \approx 10^{20} \quad , \quad \text{(b)}\ \frac{c/H_o}{l_{\mathbb{P}}} \approx 10^{60} \quad , \quad \text{(c)}\ \frac{M}{m} \approx 10^{80} \,, \qquad (1)$$

where $m_{\mathbb{P}} = (\hbar c/G)^{1/2}$ is the Planck mass, $l_{\mathbb{P}} = (\hbar G/c^3)^{1/2}$ is the Planck length, m is a typical elementary particle mass and M is the typical observed mass in the universe. From (1a) and the fact that $e^2 \approx \hbar c$, the ratio of the electric force over the gravitational force for an elementary particle of charge unity is $e^2/Gm^2 \approx 10^{40}$. Equations (1a) and (1b) are often combined to be written as $m \approx (\hbar^2 H_o / Gc)^{1/3}$. The trouble is that the Hubble "constant" varies with time: this led Dirac to propose his cosmology with varying constants, guided as he was by the idea that these relations describe "fundamental though as yet unexplained truths".[3] But observations and experiments have now ruled out variations of constants large enough to account for these relations.[8,9] Their current interpretation is in terms of 'anthropic principle' considerations.[6,10]

(ii) The third ('Mach-Einstein's principle') requires that the *inertial forces* themselves be determined by the gravitational field of the whole universe.[4-6] Except for the unrealistic Einstein's model, only some particular cosmological models are Machian in this sense, namely the flat models ($\Omega_{tot} = \Omega_M + \Omega_\Lambda = 1$, where $\Omega_M = 8\pi G\rho/3H^2$ and $\Omega_\Lambda = \Lambda c^2/3H^2$). Indeed, as shown by Sciama,[5] the implementation of this principle requires ascribing inertia to an *inductive* effect of distant matter. This implies that the gravitational field of the universe cancels that of local matter, or, in other words, that the total energy (inertial + gravitational) of a particle at rest with respect to the universe is zero. This condition reads $GMm/r \approx mc^2$, i.e., it is expressed by the Schwarzschild-like condition $GM/c^2r \approx 1$.[5,6] For models with a cosmological constant, this condition writes strictly $2GM/c^2r + \Lambda r^2/3 = 1$. If we take $r = c/H$, and $M = \frac{4}{3}\pi\rho(c/H)^3$, it becomes precisely the above flat model relation $\Omega_{tot} = 1$. These models are the only ones for which Ω is unvarying with time, i.e. the Mach-Sciama relation is satisfied whatever the epoch. However a serious drawback to such a solution is that inertia is observed to be highly isotropic, in agreement with Einstein's equivalence principle,[3] while the Sciama solution would imply a small but measurable effect of masses as large as that of our Galaxy and the local supercluster of galaxies.[3]

In fact, the implementation of the 'Mach-Einstein' principle would imply to reach a very profound level of physical knowlegde. Indeed, it amounts to express G in terms of the distribution of matter in the universe. But from the three fundamental constants G, $\hbar$ and c, one may construct the three natural Planck units of mass, length and time. Excluding G from the fundamental irreducible constants would finally amount to reduce the mass unit to length and time units.[15] Such a grand goal is to be contrasted with the present stage of physics, which is still unable to predict the mass *ratios* of elementary particles or the values of the fundamental coupling constants.

The aim of this *letter* is to propose a new solution to the cosmological constant problem, which naturally implements the 'Mach-Dirac' principle. Let us first recall that the cosmological constant is the inverse of the square of a length $\mathbb{L}$:

$$\mathbb{L} = \frac{1}{\Lambda^{1/2}} \quad . \tag{2}$$

The self-consistency of general relativity requires that it is an *absolute constant*, while observations impose[2] $\Omega_\Lambda \lesssim 1$, i.e. $\mathbb{L} \gtrsim c/\sqrt{3}H_o \approx 2$ Gpc. Its ratio with the Planck length defines a fundamental dimensionless number:

$$\mathbb{K} = \frac{\mathbb{L}}{l_{\mathbb{P}}} = \left(\frac{c^3}{\hbar G \Lambda}\right)^{1/2} \quad . \tag{3}$$

The present observational limit on Λ yields $\mathbb{K} \gtrsim 3 \times 10^{60}$.

Let us further describe the vacuum energy density problem. When considering the vacuum at a resolution $r \approx c\Delta t$, the Heisenberg uncertainty relations tells us that its energy is $E \approx \hbar c/r$, yielding a vacuum energy density $\rho_v \approx \hbar c/r^4$. This is a divergent quantity when $r \to 0$, so that one usually introduces a cutoff at the Planck scale which yields the 10^{120} too large result quoted at the beginning of this letter. However, as noticed by

Zeldovich,[11] the zero point energies themselves are unobservable : only the quantum fluctuations have a physical meaning. Zeldovich suggested that the actual vacuum energy is given by the *gravitational energy of the virtual particle-antiparticle pairs* which are continuously created and annihilated in the vacuum. This energy reads $Gm^2(r)/r$, where $m(r)$ is the effective mass at scale r, i.e. $m(r) \approx \hbar/cr$. This yields an energy density

$$\rho_V = \rho_{\mathbb{P}} \left(\frac{l_{\mathbb{P}}}{r} \right)^6 \qquad (4)$$

still equal to the Planck density at the Planck scale.

Our proposal is that, as is already the case of several quantities in quantum field theories, the vacuum energy density is an *explicitly scale-dependent quantity*. For example the electric charge is known to increase when the length-scale decreases below the Compton length of the electron, as a result of vacuum polarization by virtual particle pairs.[16] Though its small scale (‘bare’) value is much higher than its Bohr scale value (it is even formally divergent in today’s QED), we know that the bare charge is irrelevant for low energy processes where it has fallen to its classical value. We suggest that the same is true of the vacuum energy density, since the vacuum is indeed characterized by its scale-invariance (in geometrical terms, one would speak of its fractal character).[12-14]

So let us describe this scale dependence by a renormalization group equation

$$\frac{d\rho}{d\ln r} = \gamma(\rho) \ . \qquad (5)$$

The γ function is *a priori* unknown, but may be expanded, provided $\rho << 1$, to first order about the origin as $\gamma(\rho) = \gamma_0 + \gamma_1 \rho$. Then Eq. (5) is solved as

$$\rho = \rho_o \left[1 + \left(\frac{r_o}{r} \right)^{-\gamma_1} \right] \ . \qquad (6)$$

where we have set $\rho_o = -\gamma_0/\gamma_1$, and where r_o is an integration constant. The comparison with Eq. (4) yields $\gamma_1 = -6$. However the truly remarkable result here is that, when keeping the zero-order term γ_0 in the γ function, not only a power law scale-dependence is obtained, but also the transition (about scale r_o) to scale-independence at large length-scales. Equation (6) tells us that the vacuum energy density at infinity (i.e. the truly *cosmological* constant) is given by $\rho(\infty) = \rho_o$, but also that $\rho(r_o) = 2\rho_o$. Then the cosmological vacuum energy density is given, up to a factor of 2, by its value at the transition scale as derived from the asymptotic formula.

Such a behaviour is understandable is one considers that the total cosmological constant is indeed the sum of a geometrical constant term issued from general relativity, and of a quantum, scale-dependent, term due to the virtual pair self-energies, i.e., $\Lambda = \Lambda_G + \Lambda_Q(r)$. Then Eq.(6) can be understood as telling us that the matching of these two contributions naturally occurs at the scale where the quantum contribution ends.

Now the computation of the cosmological constant amounts to the determination of the scale r_o. A first lower limit on Λ is given by the Compton scale of the electron $\lambda_e = \hbar/mc$, since it is only for energy scales larger than $\approx 2m_e c^2$, i.e. for length scales smaller than $\approx \lambda_e/2$ that the phenomenon of virtual pair creation and annihilation begins to occur. However, since we are concerned here with the internal self-energies of the electron and positron of the pair, its typical scale is rather given by the e^+e^- annihilation cross-section, which reads in terms of the energy scale (to lowest order)[16]

$$\sigma(e^+e^-) = \pi\, r_e^2 \left(\frac{m_e c^2}{E}\right), \tag{7}$$

where $r_e = \alpha\, \lambda_e$ is the 'classical radius' of the electron and $\alpha \approx 1/137.036$ is the fine structure constant. This formula can be interpreted as meaning that, in the quantum theory, the effective 'radius' of the electron (which is a scale-dependent quantity $r(E)$) is precisely its classical radius at its own mass scale, namely $r(m_e c^2) = r_e$. So we propose that the transition scale r_o can be identified with the scale r_e. This suggestion is reinforced by the fact that the scale r_e seems indeed to play a remarkable role in particle physics: it corresponds to an energy of 70.02 MeV, while the effective mass of quarks in the lightest meson ($\pi^{\pm}$) is $\approx m_\pi/2 = 69.78$ MeV and, more importantly, the Quantum-Chromo-Dynamical scale is found to be $\Lambda^{(6)}_{\mathrm{QCD}} = 66 \pm 10$ MeV, for 6 quark flavours and the recently improved value of the QCD coupling at the Z boson scale,[17] $\alpha_s(m_Z) = 0.112 \pm 0.003$. Such a scale may also correspond to the end of the quark-hadron transition.

Inserting the value of r_e in Eq. (4), we find that the cosmological constant is $\Lambda = (\mathbb{K} l_{\mathbb{P}})^{-2}$, with the fundamental pure number $\mathbb{K}$ given by the relation:

$$\alpha \, \frac{m_{\mathbb{P}}}{m_e} = \mathbb{K}^{1/3} \quad . \tag{8}$$

Note that the same result is obtained by remarking that the existence of the fundamental large scale $\mathbb{L}$ implies the existence of a very small characteristic energy $E_{\min} = \hbar c/\mathbb{L}$, and by subsequently assuming that the gravitational self-energy of the electron at scale r_e equals precisely $E_{\min}$.

Equation (8) yields a precise estimate for the fundamental number $\mathbb{K}$:

$$\mathbb{K} = (5.3 \pm 2) \times 10^{60} \quad , \tag{9}$$

which agrees with the current observational limit $\mathbb{K} \gtrsim 3 \times 10^{60}$ and provides us with an explanation for one of the Dirac-Eddington large number coincidences (Eqs.1 a and b). Note that the quoted error corresponds to an uncertainty by a factor of ≈ 2 on the vacuum energy density, which may come from the uncertainty on threshold effects at the transition scale, and from the uncertainty of the transition scale itself (r_e?, $\Lambda^{(6)}_{\mathrm{QCD}}$?); but if we let

r_e be as large as $\approx\lambda_c/2$, $\mathbb{K}$ would reach 2.5 10^{64}. To the estimate of Eq. (9) there corresponds a value of the cosmological constant:

$$\Lambda \;=\; \frac{1}{\mathbb{L}^2} \;=\; 1.36 \times 10^{-56}\ \mathrm{cm}^{-2} \;, \tag{10}$$

i.e., $\Omega_\Lambda = \Lambda c^2/3H_o^2 = 0.36\,h^{-2}$ (where $h = H_o$ / 100 km/s.Mpc). Such a value would help solve the problem of the age of the universe. For example if $k=0$, it becomes larger than the observational limit[2] $\approx$13 Gyr provided $h <$ 0.75. This condition is relaxed to $h \lesssim 0.85$ if Ω_{tot} is allowed to be less than 1. In this case Ω_M would be low enough ($\lesssim$ 0.1) to be accounted for by baryons only, without disagreement with primordial nucleosynthesis.

Let us end this letter by remarking that the Sciama relation may also be implemented in a new (time-independent) way by requiring that the universe reaches its black hole horizon at length scale $\mathbb{L}$ rather than c/H. This can be done in *every* Robertson-Walker dust models, since their prime integral $\rho R^3 = cst$ allows one to define a *constant* characteristic mass:

$$\mathbb{M} = \frac{3\kappa}{8\pi} \quad \frac{c^3}{GH} \quad \Omega_M \left(\frac{k}{\Omega_M+\Omega_\Lambda-1}\right)^{3/2} \quad , \tag{11}$$

where κ depends on the geometry (for example $\kappa=2\pi^2$ if $\mathbb{M}$ is the *total* mass of a spherical model). The black hole conditions now reads $2G\mathbb{M}/c^2\mathbb{L}$ + $\Lambda\mathbb{L}^2/3 = 1$, which becomes (since $\Lambda\mathbb{L}^2 = 1$)

$$\frac{3\,G\,\mathbb{M}}{c^2\,\mathbb{L}} \;=\; 1 \quad . \tag{12}$$

Even though combining Eqs. 8 and 12 allows us to account for the second large-number coincidence (Eqs. 1b and 1c):

$$\frac{\mathbb{M}}{m_e} \;=\; \frac{\mathbb{K}^{4/3}}{3\alpha} \quad , \tag{13}$$

this does not mean that the 'Mach-Einstein principle' is implemented either, since the question of the origin of the high isotropy of inertia remains asked.

A more detailed account of this approach, along with a development of new proposals concerning the physical meaning of the Planck scale[13,14] and of the cosmic scale $\mathbb{L}$,[14] will be given in a forthcoming work.

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