Scale-relativity and quantization of planet obliquities

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Chaos, Solitons and Fractals, 9, 1035-1041, 1998

Abstract

We apply the theory of scale relativity [1], [2], [3] to the equations of rotational motion of solid bodies. We predict in the new framework that the obliquities and inclinations of planets and satellites in the solar system must be quantized. Namely, we expect their distribution to be no longer uniform between 0 and π, but instead to display well defined peaks of probability density at angles $\theta_{\mathrm{k}} = k\pi/n$. We show in the present letter that the observational data agree very well with our prediction for $n = 7$, including the retrograde bodies and those which are heeled over the ecliptic plane. In particular, the value $23°27'$ of the obliquity of the Earth, which partly determines its climate, is not a random one, but lies in one of the main probability peaks at $\theta = \pi/7$.

1 Introduction

The theory of scale relativity is founded on the giving up of the implicit hypothesis of the differentiability of space-time. Let us briefly recall the main steps of its construction. A more detailed account can be found in the book [1], in the more recent review paper [2] and in references therein.

(i) We give up differentiability of space-time coordinates while keeping their continuity. This implies their explicit dependence on resolutions. In a fractal space-time, the various physical quantities, then the equations of physics, become scale-dependent.

(ii) We re-interpret resolutions as essential, intrinsic variables that characterize the relative state of scale of the reference system, in the same way as velocities characterize its state of motion.

(iii) We extend Einstein's principle of relativity and principle of covariance, in order to include the new scale transformations.

(iv) The scale laws to be constructed are found to combine a standard fractal (power-law) behavior at small and large scales, and a transition to scale-independence at intermediate scales (scale symmetry breaking). In other words, space-time, that is Riemannian at intermediate scales, becomes fractal toward very small and very large scales.

(v) In the fractal domains, scale-covariance transforms classical mechanics into a quantum-like mechanics. In what follows, we shall apply this result to the rotational motion of a solid body.

Scale relativity, when combined with the laws of gravitation, provides us with a general theory of the structuring of gravitational systems. We have already shown [1], [4], [5] that the theory accounts in a very constrained way for several structures observed in the Solar System, including planet distances, eccentricities, and mass distribution. More recently, we have demonstrated that it also applies to all extra-solar planetary systems, in terms of the same universal constant as in our own Solar System [3]. In particular, the system of three planets discovered around the pulsar PSR B1257+12 agree with our prediction with a very high precision of some 10^{-4} [3], [6]. The theory also explains the various Tifft's effects [7], [8] of redshift quantizations in the galactic and extragalactic domains [2], [6], [9] and predicts new quantization effects [10].

Now the ability of the theory to predict structures (in terms of peaks of probability density) must be true not only for translational motion, but also for rotational motion. It is the aim of the present letter to check this claim by applying the theory to the distribution of obliquities and inclinations in the Solar System. We predict that this distribution must show peaks of probability density distributed in a periodical way, then we show that the available data agree with this theoretical prediction.

2 Theory

The classical equations of motion of a solid body can be given the form of Euler-Lagrange equations, and therefore come in a very easy way under our theory, since our "quantization" procedure applies in a general way to these equations [1]. We adopt throughout this section the tensorial notation (a summation is meant on all couples of identical indices).

Let us first recall how the problem is posed in classical mechanics. The role of the variables (x, v, t) is now played by (φ, Ω, t), where φ stands for the three rotational Euler angles and Ω for the angular velocity. The Lagrange equation writes :

$$\frac{d}{dt}\frac{\partial L}{\partial \Omega} = \frac{\partial L}{\partial \varphi}, \tag{1}$$

with respect to the rotational coordinates. The Lagrange function writes:

$$L = \frac{1}{2} m v^2 + \frac{1}{2} I_{\rm ik} \Omega_{\rm i} \Omega_{\rm k} - U, \tag{2}$$

where $I_{\rm ik}$ is the tensor of inertia of the body. In general, one choses for simplicity to define the rest frame by the principal axes of inertia of the solid body. In this reference frame, the tensor of inertia is diagonal, $I_{\rm ik} \rightarrow I_{\rm k}$. For a solid body, the right-hand member of Eq. 1 writes:

$$\frac{\partial L}{\partial \varphi} = -\frac{\partial U}{\partial \varphi} = K = \sum r \times F, \tag{3}$$

which identifies with the total torque, i.e., the sum of the moments of all forces acting on the body [11]. In the left-hand member one recognizes the angular momentum about the center of mass,

$$M_{\rm k} = \frac{\partial L}{\partial \Omega_{\rm k}} = I_{\rm kj} \; \Omega_{\rm j}, \tag{4}$$

and we finally recover a rotational equation of dynamics similar to Newton's:

$$I_{\rm jk} \, \frac{d\Omega_{\rm k}}{dt} = K_{\rm j}. \tag{5}$$

In the scale relativity framework, the above classical mechanics equations become wrong on large time scales. We shall now see that they must be replaced by quantum-mechanical like equations (whose interpretation is, however, partly different from that of standard quantum mechanics).

Let us assume that the rotational motion of the solid body under consideration is highly chaotic, either because of external perturbations by the environment, or because of the fractal and non-differentiable geometry of space-time (that plays the role of a universal perturbation), or both, and let us place ourselves on time-scales large with respect to the chaos time. We are in the same conditions as in the theory of translational motion, but now the position angles have replaced the coordinates, and the tensor of inertia have replaced the mass, which implies a generalization of our equations. In our nondeterministic approach, we definitively give up the hope to make strict predictions about the values of these angles, and we now work in terms of probability amplitude for these values. By this way we become able to predict structures, since all values of the angles will no longer be equivalent, but instead some of them will be favored, corresponding to peaks of probability density.

Following the same road as for position coordinates, we describe the effect on angles of the fractal structure of space-time in terms of fractal fluctuations of dimension 2 and two-valuedness of the angular velocity [1],[2]. That leads to introducing a complex angular velocity $\tilde{\Omega}$, then a complex Lagrange function $\tilde{L}(\varphi, \tilde{\Omega}, t)$. The two effects of the non-differentiability and fractality of space can be combined in terms of a scale-covariant derivative acting on the *mean* angles (for which differentiability is preserved):

$$\frac{d\!\!\!/}{dt} = \frac{\partial}{\partial t} + \tilde{\Omega}_{\rm k} \cdot \partial_{\rm k} - i\,\mathcal{D}_{\rm jk}\partial_{\rm j}\partial_{\rm k}, \tag{6}$$

where the indices j and $k = 1$ to 3 run on the three rotational Euler angles and where $\mathcal{D}_{\rm jk}$ is a fundamental rotational tensor which characterizes the effect of the fractal geometry of space-time, and can be, in the first approximation, identified with a diffusion matrix.

The quantization of Eq. 5 is straighforward using this scale-covariant derivative. It writes:

$$I_{\rm jk}\,\frac{d\!\!\!/\,\tilde{\Omega}_{\rm k}}{dt} = -\partial_{\rm j} U. \tag{7}$$

We finally introduce a complex probability amplitude as another expression for the complex action $S = \int \tilde{L} dt$:

$$\psi = e^{iS/S_0}, \tag{8}$$

where S_0 is introduced for dimensional reasons. Provided the constant S_0 be given in terms of $\mathcal{D}_{\rm jk}$ and $I_{\rm jk}$ by the relation

$$S_0 \;= 2\,I_{\rm jk}\mathcal{D}_{\rm jk}\,, \tag{9}$$

(which generalizes the relation $S_0 = 2m\mathcal{D}$ obtained in the case of translational motion [1]), Eq. 7 can be integrated to yield a Schrödinger-like equation acting on Euler angles, in terms of the probability amplitude ψ:

$$S_0(\mathcal{D}_{\rm jk}\partial_{\rm j}\partial_{\rm k}\psi + i\,\frac{\partial}{\partial t}\psi) = U\,\psi. \tag{10}$$

The physical meaning of this equation can be specified by writing its imaginary part in the form of a continuity equation. This can be done by first jumping to the reference system where $\mathcal{D}_{\rm jk}$ is diagonal, i.e., $\mathcal{D}_{\rm jk} = \mathcal{D}_{\rm k}$, then by performing a new change of angle coordinates, introducing reduced angles $d\zeta_{\rm k} = d\varphi_{\rm k}/(\mathcal{D}_{\rm k})^{1/2}$. In terms of these reduced angles and of the corresponding rotational velocities, the imaginary part of Eq. 10 becomes an equation of continuity, in which the probability density is given by the square of the modulus of the probability amplitude, $P(\zeta_{\rm k}) = \psi\psi^\dagger$. As in the case of translational motion, we shall interpret the existence of peaks of probability density (which are consequences of the shape of the potential and/or of the limiting conditions) as a tendency for the system to make structures (here, angular structures).

We shall now apply this equation to our own Solar System, and show that its solutions allow one to explain some observed characteristics of the distribution of planet obliquities and of the inclination of their orbits.

3 Application to planet obliquities

Let us look for the solutions of our generalized Schrödinger equation in the simplest case. Namely, being interested here in obliquities and inclinations, we treat either the planet or satellite, or the couple planet/satellite-orbit as a spinning top. We shall consider in what follows only the problem of free rotational motion, focusing on the first Euler angle θ. It is only in that particular and simplified case that obliquities and inclinations can be treated together as solutions of the same equation. The scale-relativistic equation of evolution of the angle θ writes:

$$\frac{\acute{d}^2\theta}{dt^2} = 0. \tag{11}$$

Looking for stationary solutions, Eq. 11 takes after integration the form of a one-dimensional time-independent Schrödinger equation:

$$\frac{d^2\psi}{d\theta^2} + A^2\psi = 0, \tag{12}$$

where $A = E/2I = \Omega/2\mathcal{D}$ since the rotational energy is $E = \frac{1}{2}\, I\, \Omega^2$.

The solutions which have a peak at $\theta = 0$ (as observed and as predicted by the classical theory) and which satisfy the periodic condition $P(\theta + \pi) = P(\theta)$, write:

$$P(\theta) = a\, \cos^2(n\theta) \tag{13}$$

with n integer.

4 Comparison with observational data

The peaks in the observed distribution of both obliquities and inclinations of planets and satellites in the Solar System agree remarkably well with the predicted periodicity of Eq. 13, in terms of a unique value $n = 7$ for the whole system. This can be seen in Figs. 1 and 2 (the data is taken from Beatty and Chaikin [12], Encrenaz et al. [13] and Lang [14]; we mix values of the inclination relative to the planet equator with those relative to the orbital plane, since a common quantization law is predicted for all these values). Note that, at the still rough level of our analysis (free rotational motion description) our prediction concerns only the periodicity of the most probable values of the angles, but not the amplitude of the peaks. This result allows one to explain some very striking features shared by the observed inclinations and obliquities in our Solar System, in particular:

(i) The value $23^\circ.45$ of the obliquity of the Earth is considered in the classical description to be the chance result of the last collision at the end of the formation epoch. This collision has probably given rise to the Earth-Moon couple, then

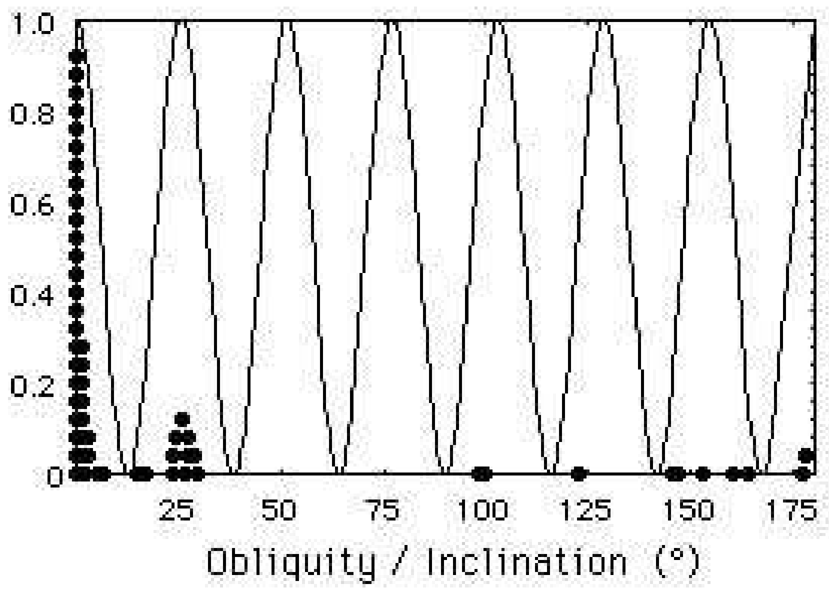

Figure 1: Comparison between the observed obliquities and inclinations of planets and satellites in the solar system (black points: some of them have been displaced vertically for the readibility of the diagram) with the periodicity predicted by the scale-relativistic approach. The peak in $\theta = 0$ is already expected in the classical theory.

the Earth obliquity has been stabilized to its value by the Moon [15]. Moreover, all of the terrestrial planets have probably experienced large, chaotic variations of their obliquities at some time in the past. In particular the obliquity of Mars is still in a large chaotic region [16]. This classical description is probably globally correct, but cannot explain why a large number of bodies in the Solar System have values of their obliquities and inclinations very close to the Earth value (see Fig. 1). Indeed the obliquities of Mars, Saturn and Neptune are respectively 23.98°, 26.73°, and 29.6°, (mean $25.9 \pm 1.4°$ for 4 points), and the inclinations of Jupiter's satellites Leda, Himalia, Lysithea and Elara, and of Neptune's satellite Nereid are respectively 26.1°, 27.6°, 28.8°, 24.8° and 27.5° (full mean $26.5 \pm 0.7°$ for 9 points). Both average values lie within one sigma of our first non-zero predicted probability peak at $180°/7 = 25.7°$ (see Fig. 1).

(ii) The obliquity of Uranus, which is heeled over its orbital plane, is actually 98°, within 5° of the predicted peak at $\theta = 102.8°$ ($k = 4$). The inclination $\approx 100°$ of Charon's orbit around Pluto also agrees with this peak.

(iii) The retrograde rotations of Venus on itself ($\theta = 178°$), which is one of the puzzling feature of the Solar System [16] and of Saturn's satellite Phoebe on its orbit ($\theta = 177°$) also come under our prediction, since they agree with the probability peak at 180°. The same is true of the orbital inclination of four of the Jupiter satellites and of Neptune's satellite Triton, which yield a mean value $153.6° \pm 3.6°$, to be compared with the predicted peak $k = 6$ at $\theta = 154.3°$.

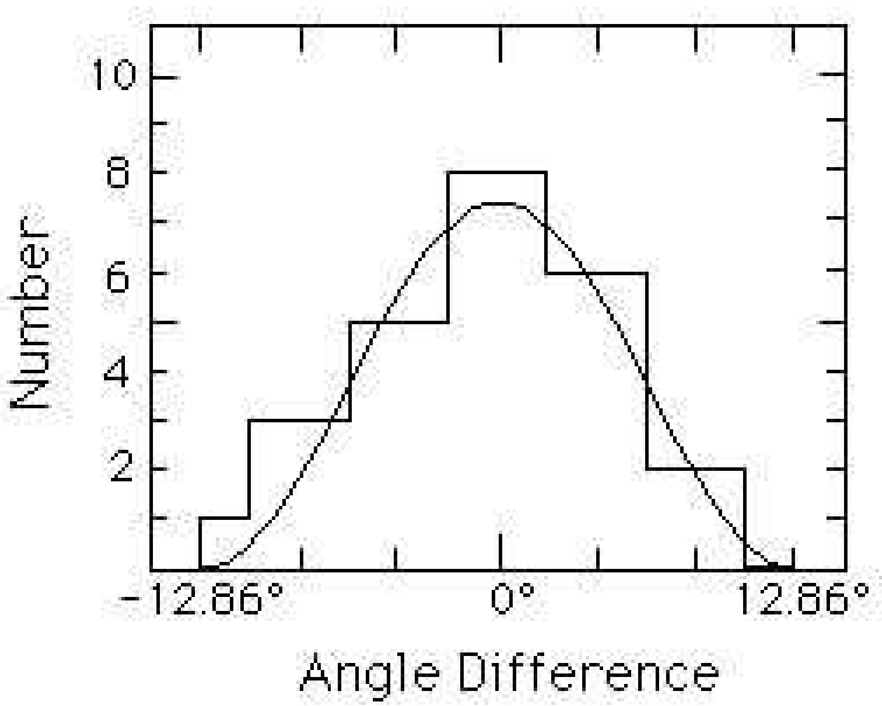

Figure 2: Distribution of the differences between the observed obliquities and inclinations of planets and satellites in the solar system and the nearest predicted probability peak (given by $k\pi/7$) for $k \neq 0$. There is an additional sharp peak at zero obliquity and inclination (33 points), which is excluded from this plot for its readibility. From a χ^2 test, the probability to obtain such a discrepancy with a uniform distribution is $P \ll 10^{-4}$. The observed distribution is then compared with our theoretical prediction (sine curve).

5 Discussion and conclusion

Before concluding, it may be useful to be more specific about the physical meaning of our theory and of our results. We stress once again the fact that, although some of the mathematical tools of scale relativity are in common with standard quantum mechanics (probability amplitude, Schrödinger equation), we do not recover the whole quantum mechanical interpretation when scale relativity is applied to macrophysics as in the present work. In particular we do not expect the Bell inequalities to be violated. Indeed, contrarily to what happens in the microphysical case, the differentiability of coordinates is recovered at small time scales, for which, as a consequence, the classical theory still holds.Therefore our macroscopic quantum-like theory is a kind of "hidden parameter" theory, contrarily to the standard quantum mechanics.

In this regard, we do not think that our results contradict those obtained from standard celestial mechanics, which are valid below the prediction horizon, while our theory applies beyond the horizon. Both approaches are therefore complementary. Peaks of probability such as those we predict can be obtained from chaotic classical motion (case of Mars) at small time scales if, e.g., the time elapsed in the peaks is far larger than the time needed to jump from one peak to another. In the case of the Earth obliquity, the argument of its stabilization by the Moon [15] remains valid, but our theory adds a prediction of the value at which it can be stabilized (namely, around 25°).

Recall also that our results are obtained in the framework of a very simplified description (free rotational motion). It has however already predictive power: Two probability peaks at 54.4° and 77.1° are still empty and could be filled by future observations. Hence the peak at 128.6° was empty until the recent measurement of Pluton's obliquity at 122.5°. In works to come, we shall try to improve our model and to answer questions that remain open, concerning in particular the origin of the empirical value $n = 7$.

Let us conclude by remarking that our theory reinforces greatly the probability that a large number of Earth-like planets exist in the universe, stable enough for life to develop and survive. Indeed, we predict that around solar-type stars, the orbit $n = 5$ will always have a semi-major axis of ≈ 1 AU and be nearly circular [1], [3], [5], and that its obliquity can be $\approx 25°$ with high probability.

Acknowledgements: I gratefully acknowledge help from Dr. E.T. Lefèvre in the processing of this paper.

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