# Numerical relativity and the simulation of gravitational waves

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# Gravitational waves: an introduction

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# Gravitational waves

## definition

Gravitational waves are predicted in Einstein's relativistic theory of gravity: general relativity

**Einstein's equations**

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Linearizing around the flat (Minkowski) solution in vacuum $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$:

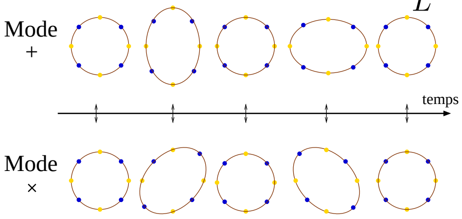
$$\Box \left( h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \right) = -16\pi T_{\mu\nu}.$$

# Gravitational waves

## effects and amplitudes

The effect of a gravitational wave of (dimensionless) amplitude $h$ is a brief change of the relative distances

$$\frac{\Delta L}{L} \sim h.$$




Two polarization modes "+" and "×": corresponding to the two dynamical degrees of freedom of the gravitational field.

Using the linearized Einstein equations:
⇒at first order $h \sim \ddot{Q}$ (mass quadrupole momentum of the source), the total gravitational power of a source is

$$L \sim \frac{G}{c^5} s^2 \omega^6 M^2 R^4.$$

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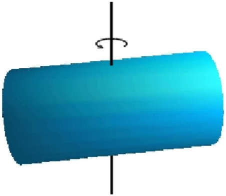
# Gravitational waves

## a laboratory experiment?

The proof of the existence of electromagnetic waves has been achieved by producing them in a laboratory and detecting them. Can we do this with gravitational waves?

- electromagnetic waves are produced by accelerating electric charges,
- gravitational waves are produced by accelerating masses.

Trying to accelerate a mass by rotating it



Consider a cylinder made of steel

- one meter in diameter and **twenty** meters long,
- weighting about **490 tons**,
- rotating at a maximal velocity of 260 rotations/minute (before breaking apart),

⇒**ABSOLUTELY NO HOPE** of detection, the emission is much too low.

# Gravitational waves

## astrophysical sources

The problem stems from the constant factor in

$$L \sim \frac{G}{c^5} s^2 \omega^6 M^2 R^4$$

Introducing the Schwarzschild radius (radius of a black hole having the same mass) $R_S = \dfrac{2GM}{c^2}$, one gets

$$L \sim \frac{c^5}{G} s^2 \left(\frac{R_S}{R}\right)^2 \left(\frac{v}{c}\right)^6$$

⇒accelerated masses:

- with strong gravitational field $\iff$ compact: neutron stars & black holes,
- at relativistic speeds,
- far from spherical symmetry ($s \lesssim 1$).

Binary systems of compact objects, neutrons stars & supernovae.

# Gravitational waves

## ground detectors

### LIGO: USA, Louisiana

### LIGO: USA, Washington

### VIRGO: France/Italy near Pisa

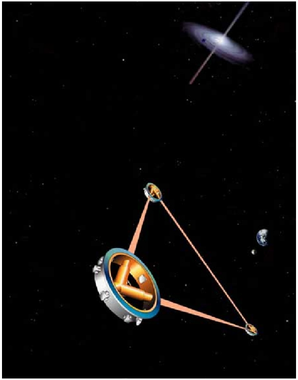
Michelson-type interferometers with 3 km (VIRGO) and 4 km (LIGO) long arms and almost perfect vacuum! Frequency range 10 $\rightarrow$ 10000 Hz.

$\Rightarrow$Have been acquiring data together since a couple of years.

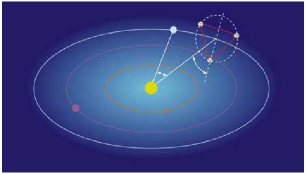
# Gravitational waves

## space project LISA

On Earth, the vibrations propagating on the crust (seismic noise, human activities, …) are limiting the detectors' sensitivity.



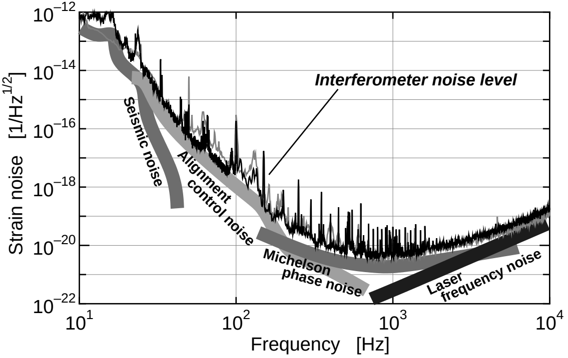
⇒LISA project (ESA / NASA) should be launched in 2019: 3 satellites at 5 millions kilometers one from another, in orbit around the Sun, 20 degrees behind the Earth. Frequency range $10^{-4} \rightarrow 1$ Hz.



Many more sources to be detected, with even a few certain ones.

# Gravitational waves

## compute waveforms!




- The signal at the output of the detector $\sigma(t) = h(t) + n(t)$, with $h(t) \leq n(t)$.

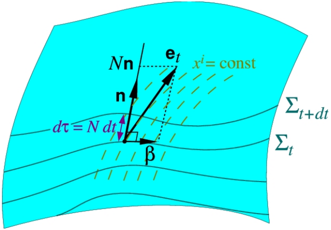
- The probability of detection is greatly enhanced in case of matched filtering: convolution with *a priori* known signal.

$\Rightarrow$ Need of full database of possible waveforms, to be computed by any means: analytic (post-Newtonian, ...) or numeric (our group).

# Formulation of Einstein equations

# 3+1 FORMALISM

Decomposition of spacetime and of Einstein equations




**EVOLUTION EQUATIONS:**

$$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}} K_{ij} =$$
$$-D_i D_j N + N R_{ij} - 2N K_{ik} K^k{}_j +$$
$$N \left[ K K_{ij} + 4\pi((S-E)\gamma_{ij} - 2S_{ij}) \right]$$
$$K^{ij} = \frac{1}{2N}\left( \frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i \right).$$

**CONSTRAINT EQUATIONS:**

$$R + K^2 - K_{ij}K^{ij} = 16\pi E,$$
$$D_j K^{ij} - D^i K = 8\pi J^i.$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -N^2\, dt^2 + \gamma_{ij}\,(dx^i + \beta^i dt)\,(dx^j + \beta^j dt)$$

# Constrained / free formulations

As in electromagnetism, if the constraints are satisfied initially, they remain so for a solution of the evolution equations.

**Free evolution**

- start with initial data verifying the constraints,
- solve only the 6 evolution equations,
- recover a solution of all Einstein equations.

⇒apparition of constraint violating modes from round-off errors. Considered cures:

- Using of constraint damping terms and adapted gauges.
- Solving the constraints at every time-step: e.g. fully-constrained formalism in Dirac gauge (2004).

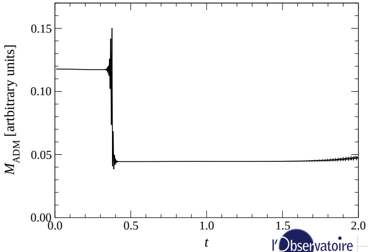
# Conformal-flatness condition

## Uniqueness issue

4 constraints and the choice of time-slicing (gauge)
⇒elliptic system of 5 non-linear equations can be formed

- Elliptic part of Einstein equations in the constrained scheme,
- Conformal-Flatness Condition (CFC): no evolution, no gravitational waves. used for computing initial data.

Because of non-linear terms, the elliptic system may not converge ⇒the case appears for dynamical, very compact matter and GW configurations (before appearance of the black hole).



# Summary of Einstein equations

## constrained scheme

### EVOLUTION

$$
\begin{aligned}
\frac{\partial A^{ij}}{\partial t} &= \nabla^k \nabla_k \tilde{\gamma}^{ij} + \dots \\
\frac{\partial \tilde{\gamma}^{ij}}{\partial t} &= 2N\Psi^{-6} A^{ij} + \dots
\end{aligned}
$$

with

$$
\begin{aligned}
\det \tilde{\gamma}^{ij} &= 1, \\
\nabla_j^{(f)} \tilde{\gamma}^{ij} &= 0.
\end{aligned}
$$

### CONSTRAINTS

$$
\begin{aligned}
\nabla_j A^{ij} &= 8\pi \Psi^{10} S^i, \\
\Delta \Psi &= -2\pi \Psi^{-1} E \\
&\quad -\Psi^{-7} \frac{A^{ij} A_{ij}}{8}, \\
\Delta N \Psi &= 2\pi N \Psi^{-1} + \dots
\end{aligned}
$$

$$A^{ij} = \Psi^{10} K^{ij}$$

with

$$\lim_{r\to\infty} \tilde{\gamma}^{ij} = f^{ij}, \lim_{r\to\infty} \Psi = \lim_{r\to\infty} N = 1.$$

# Spectral methods
# for numerical relativity

# Simplified picture

(see also Grandclément & JN 2009)

How to deal with functions on a computer?

$\Rightarrow$a computer can manage only integers

In order to represent a function $\phi(x)$ (e.g. interpolate), one can use:

- a finite set of its values $\{\phi_i\}_{i=0\ldots N}$ on a grid $\{x_i\}_{i=0\ldots N}$,
- a finite set of its coefficients in a functional basis $\phi(x) \simeq \sum_{i=0}^{N} c_i \Psi_i(x)$.

In order to manipulate a function (e.g. derive), each approach leads to:

- finite differences schemes

$$\phi'(x_i) \simeq \frac{\phi(x_{i+1}) - \phi(x_i)}{x_{i+1} - x_i}$$

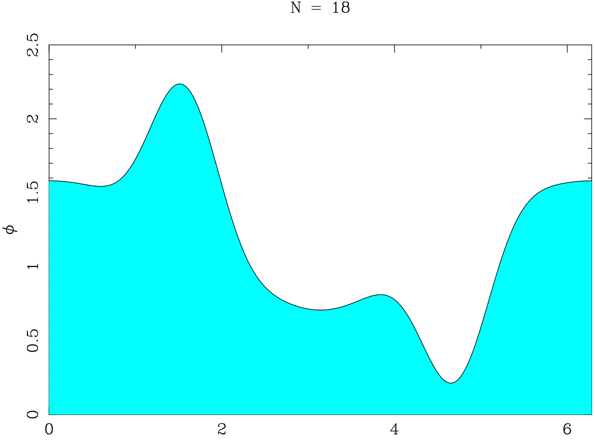
- spectral methods

$$\phi'(x) \simeq \sum_{i=0}^{N} c_i \Psi_i'(x)$$

# Convergence of Fourier series

$$\phi(x) = \sqrt{1.5 + \cos(x)} + \sin^7 x$$

$$\phi(x) \simeq \sum_{i=0}^{N} a_i \Psi_i(x) \text{ with } \Psi_{2k} = \cos(kx),\ \Psi_{2k+1} = \sin(kx)$$



## USE OF ORTHOGONAL POLYNOMIALS

The solutions $(\lambda_i, u_i)_{i\in\mathbb{N}}$ of a singular Sturm-Liouville problem on the interval $x \in [-1, 1]$:

$$-(pu')' + qu = \lambda wu,$$

with $p > 0, \mathcal{C}^1, p(\pm 1) = 0$

- are orthogonal with respect to the measure $w$:
$$(u_i, u_j) = \int_{-1}^{1} u_i(x)u_j(x)w(x)\mathrm{d}x = 0 \text{ for } m \neq n,$$
- form a spectral basis such that, if $f(x)$ is smooth $(\mathcal{C}^\infty)$
$$f(x) \simeq \sum_{i=0}^{N} c_i u_i(x)$$
converges faster than any power of $N$ (usually as $e^{-N}$).

Gauss quadrature to compute the integrals giving the $c_i$'s. Chebyshev, Legendre and, more generally any type of Jacobi polynomial enters this category.

# Method of weighted residuals

General form of an ODE of unknown $u(x)$:

$$\forall x \in [a,b],\ Lu(x) = s(x),\ \text{and}\ \ Bu(x)|_{x=a,b} = 0,$$

The approximate solution is sought in the form

$$\bar{u}(x) = \sum_{i=0}^{N} c_i \Psi_i(x).$$

The $\{\Psi_i\}_{i=0\dots N}$ are called trial functions: they belong to a finite-dimension sub-space of some Hilbert space $\mathcal{H}_{[a,b]}$.
$\bar{u}$ is said to be a numerical solution if:

- $B\bar{u} = 0$ for $x = a, b$,
- $R\bar{u} = L\bar{u} - s$ is "small".

Defining a set of test functions $\{\xi_i\}_{i=0\dots N}$ and a scalar product on $\mathcal{H}_{[a,b]}$, $R$ is small iff:

$$\forall i = 0 \dots N, \quad (\xi_i, R) = 0.$$

It is expected that $\lim_{N\to\infty} \bar{u} = u$, "true" solution of the ODE.

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# Inversion of linear ODEs

Thanks to the well-known recurrence relations of Legendre and Chebyshev polynomials, it is possible to express the coefficients $\{b_i\}_{i=0...N}$ of

$$Lu(x) = \sum_{i=0}^{N} b_i \left| \begin{array}{l} P_i(x) \\ T_i(x) \end{array} \right. , \text{ with } u(x) = \sum_{i=0}^{N} a_i \left| \begin{array}{l} P_i(x) \\ T_i(x) \end{array} \right. .$$

If $L = \mathrm{d}/\mathrm{d}x, x\times, \dots,$ and $u(x)$ is represented by the vector $\{a_i\}_{i=0...N}$, $L$ can be approximated by a matrix.

Resolution of a linear ODE

$\Updownarrow$

inversion of an $(N+1) \times (N+1)$ matrix

With non-trivial ODE kernels, one must add the boundary conditions to the matrix to make it invertible!

# Some singular operators

$u(x) \mapsto \dfrac{u(x)}{x}$ is a linear operator, inverse of $u(x) \mapsto xu(x)$.

Its action on the coefficients $\{a_i\}_{i=0\ldots N}$ representing the $N$-order approximation to a function $u(x)$ can be computed as the product by a regular matrix.⇒The computation in the coefficient space of $u(x)/x$, on the interval $[-1, 1]$ always gives a finite result (both with Chebyshev and Legendre polynomials).

⇒The actual operator which is thus computed is

$$u(x) \mapsto \frac{u(x) - u(0)}{x}.$$

⇒Compute operators in spherical coordinates, with coordinate singularities

$$\text{e.g. } \Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\theta\varphi}$$

# Explicit / implicit schemes

Let us look for the numerical solution of ($L$ acts only on $x$):

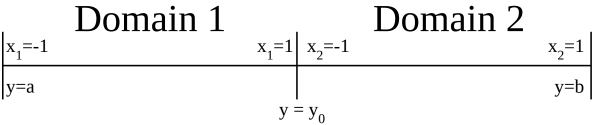
$$\forall t \geq 0, \quad \forall x \in [-1, 1], \quad \frac{\partial u(x,t)}{\partial t} = Lu(x,t),$$

with good boundary conditions. Then, with $\delta t$ the time-step: $\forall J \in \mathbb{N}, \quad u^J(x) = u(x, J \times \delta t)$, it is possible to discretize the PDE as

- $u^{J+1}(x) = u^J(x) + \delta t \, Lu^J(x)$: explicit time scheme (forward Euler); easy to implement, fast but limited by the CFL condition.
- $u^{J+1}(x) - \delta t \, Lu^{J+1}(x) = u^J(x)$: implicit time scheme (backward Euler); one must solve an equation (ODE) to get $u^{J+1}$, the matrix approximating it here is $I - \delta t \, L$. Allows longer time-steps but slower and limited to second-order schemes.

# Multi-domain approach

Multi-domain technique : several touching, or overlapping, domains (intervals), each one mapped on $[-1, 1]$.




- boundary between two domains can be the place of a discontinuity ⇒recover spectral convergence,
- one can set a domain with more coefficients (collocation points) in a region where much resolution is needed ⇒fixed mesh refinement,
- 2D or 3D, allows to build a complex domain from several simpler ones,

Depending on the PDE, matching conditions are imposed at $y = y_0 \iff$ boundary conditions in each domain.

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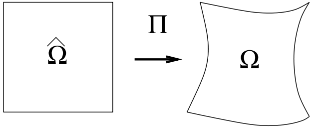
# Mappings and multi-D

In two spatial dimensions, the usual technique is to write a function as:




$$f \quad : \quad \hat{\Omega} = [-1,1] \times [-1,1] \rightarrow \mathbb{R}$$

$$f(x,y) \quad = \quad \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} c_{ij} P_i(x) P_j(y)$$

The domain $\hat{\Omega}$ is then mapped to the real physical domain, trough some mapping $\Pi : (x,y) \mapsto (X,Y) \in \Omega$.

$\Rightarrow$When computing derivatives, the Jacobian of $\Pi$ is used.

## COMPACTIFICATION

A very convenient mapping in spherical coordinates is

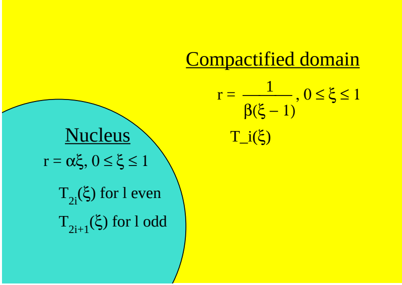
$$x \in [-1,1] \mapsto r = \frac{1}{\alpha(x-1)},$$

to impose boundary condition for $r \rightarrow \infty$ at $x = 1$.

# Example:

## 3D Poisson equation, with non-compact support

To solve $\Delta\phi(r,\theta,\varphi) = s(r,\theta,\varphi)$, with $s$ extending to infinity.




- setup two domains in the radial direction: one to deal with the singularity at $r = 0$, the other with a compactified mapping.
- In each domain decompose the angular part of both fields onto spherical harmonics:

$$\phi(\xi,\theta,\varphi) \simeq \sum_{\ell=0}^{\ell_{\max}} \sum_{m=-\ell}^{m=\ell} \phi_{\ell m}(\xi) Y_\ell^m(\theta,\varphi),$$

- $\forall(\ell, m)$ solve the ODE: $\dfrac{\mathrm{d}^2\phi_{\ell m}}{\mathrm{d}\xi^2} + \dfrac{2}{\xi}\dfrac{\mathrm{d}\phi_{\ell m}}{\mathrm{d}\xi} - \dfrac{\ell(\ell+1)\phi_{\ell m}}{\xi^2} = s_{\ell m}(\xi),$
- match between domains, with regularity conditions at $r = 0$, and boundary conditions at $r \to \infty$.

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# Application
# to binary compact stars

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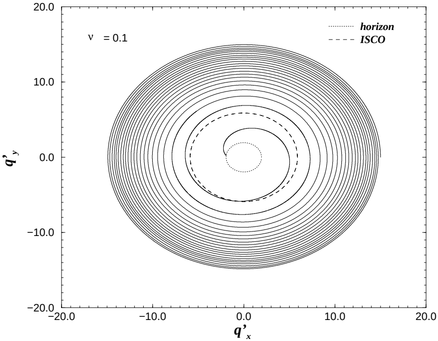
# Inspiralling binaries

Astrophysical scenario: binary systems of compact objects evolve toward the final coalescence by emission of gravitational waves and angular momentum loss.




Stiff problem: the orbital and coalescence timescales are very different.
Post-Newtonian (perturbative) computations assume point-mass particles
⇒valid until separation is comparable to size.

⇒ numerical simulation of initial data and evolution.

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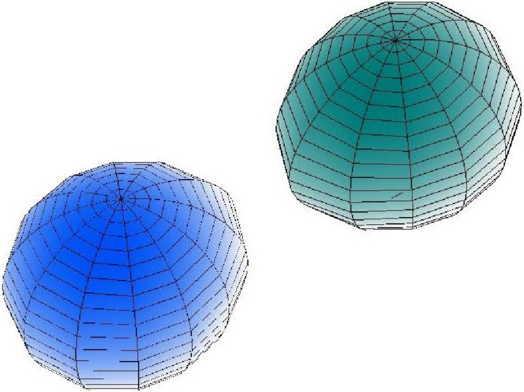
# Binary neutron stars

## binary quark stars

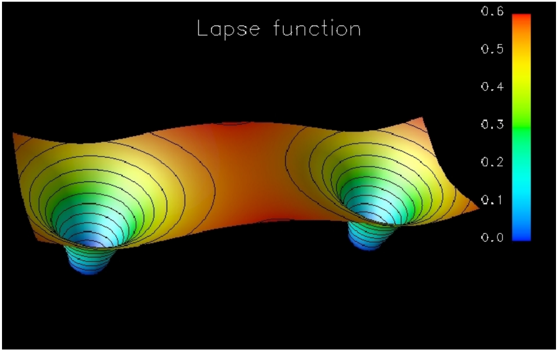
- Initial data: irrotational flow and conformal-flatness approximation,
- two adapted-grid system, to take into account tidal effects,



- use of realistic equations of state for cold nuclear matter,
- exploration of strange-quark equations of state.

# Binary black hole

Stellar masses (for VIRGO) or galactic masses (for LISA).




- First realistic initial data (2002), with excision techniques,
- Good agreement with post-Newtonian computations,
- Determination of the last stable orbit, important for gravitational wave data analysis.

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# Stellar core-collapse simulations

# Simplified physical model of core-collapse

The phenomenon of *supernova* is too rich to be fully-modeled on a computer

- relativistic hydrodynamics ($v/c \sim 0.3$), including shocks, turbulence and rotation,
- strong gravitational field $\Rightarrow$General Relativity?
- neutrino transport (matter deleptonization)
- nuclear equation of state (EOS)
- radiative transfer and ionization of higher layers
- magnetic field?

$\Rightarrow$to track gravitational waves, some features must be neglected...and we use an effective model (not trying to make them explode).

# Simplified physical model of core-collapse

- General-relativistic hydrodynamics: 5 hyperbolic PDEs in conservation form,
- Conformal-flatness condition for the relativistic gravity: 5 elliptic PDEs to be solved at each time-step,
- Initial model is a rotating polytrope with an effective adiabatic index $\gamma \lesssim 4/3$. During the collapse, when the density reaches the nuclear level, $\gamma \to \gamma_2 \gtrsim 2$,
- Passive magnetic field,
- Lepton fraction deduced from density, following spherically-symmetric simulations with more detailed neutrino transport.

# Combination of two numerical techniques

- hydrodynamics ⇒High-Resolution Shock-Capturing schemes (HRSC), also known as Godunov methods, here implemented in General Relativity;
- gravity ⇒multi-domain spectral solver using spherical harmonics and Chebyshev polynomials, with a compactification of type $u = 1/r$.

Use of two numerical grids with interpolation:

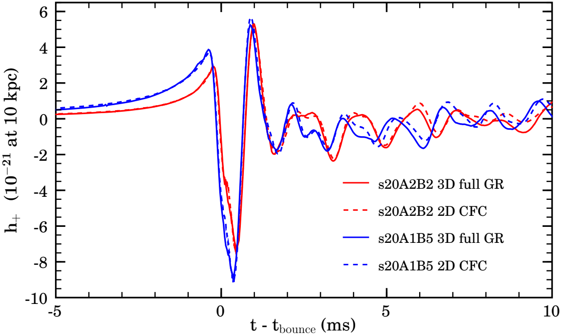
- matter sources: Godunov (HRSC) grid $\rightarrow$ spectral grid;
- gravitational fields: spectral grid $\rightarrow$ Godunov grid.

First achieved in the case of spherical symmetry, in tensor-scalar theory of gravity (JN & Ibáñez 2000).
Spares a lot of CPU time in the gravitational sector, that can be used for other physical ingredients.

# Toward a realistic relativistic collapse

Together with the use of a purely finite-differences code in full GR, first results of realistic collapse of rotating stellar iron cores in GR

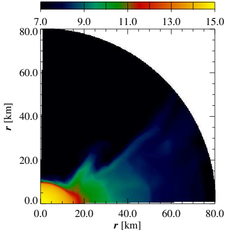
- with finite temperature EOS;
- (approximate) treatment of deleptonization.




⇒complete check that CFC is a good approximation in the case of core-collapse.

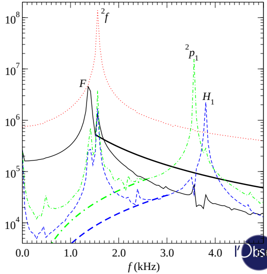
# Neutron star oscillations

Study of non-linear axisymmetric pulsations of rotating relativistic stars




- uniformly and differentially rotating relativistic polytropes ⇒differential rotation significantly shifts frequencies to smaller values;
- mass-shedding-induced damping of pulsations, close to maximal rotation frequency.

- most powerful modes could be seen by current detectors if the source is about $\sim 10$ kpc;
- if 4 modes are detected, information about cold nuclear matter equation of state could be extracted ⇒gravitational asterosismology.



# Summary – Perspectives

- Numerical simulations of sources of gravitational waves are of highest importance for the detection
- Use of spectral methods can bring high accuracy with moderate computational means (exploration of parameter space)
- Spectral methods can be associated with other types, as in the core-collapse code presented here
- Core-collapse code: going beyond conformal-flatness approximation $\Rightarrow$better extraction of waves
- Improvement of this code: realistic EOS, temperature effects for very massive star collapses (hypernovae). Neutrinos? Ongoing work with M. Oertel
- Study of the electro-weak processes: electron capture rate, nucleon effective masses and EOS. Work by A. Fantina, P. Blottiau, J. Margueron, P. Pizzochero, . . .