The gravitational field equations in CoCoNuT

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Introduction:
The need for relativistic gravity

Core Collapse using Nu Technologies

- Evolution of self-gravitating stellar bodies: degenerate stellar cores, neutron stars and black holes.
- Need to model fluid evolution (hydrodynamics) and gravitational interaction;
- Newtonian or relativistic?

⇒Black Holes: obviously relativistic!!

⇒Neutron stars: compaction parameter $\Xi = \dfrac{2GM}{Rc^2} \sim 0.2$, existence of maximal mass...

⇒What about core collapse?

Study by Dimmelmeier *et al.* (2006)

Model	*Method N*		*Method R*		*Method A*		*Method CFC*		*Method CFC+*		GR
M5a1	3.8	(0.58)	6.1	(0.92)	5.6	(0.85)	6.6	(1.00)	6.6	(1.00)	6.6
M5c2	1.1	(0.22)	3.3	(0.66)	2.9	(0.58)	4.9	(0.98)	4.9	(0.98)	5.0
M7a4	5.6	—	—	—	14	—	14	—	14	—	—
M7b1	0.10	(0.13)	0.40	(0.51)	0.31	(0.39)	0.83	(1.05)	0.85	(1.08)	0.79
M7c3	1.2	(0.13)	5.3	(0.58)	4.2	(0.46)	9.2	(1.00)	9.2	(1.00)	9.2
M8a1	4.5	—	—	—	17	—	—	—	—	—	—
M8c2	0.19	(0.04)	1.5	(0.28)	9.0	(0.17)	5.3	(0.98)	5.2	(0.96)	5.4
M8c4	1.2	(0.08)	7.1	(0.47)	5.1	(0.34)	17	(1.13)	12	(0.80)	15

Maximum density $\rho_{\mathrm{max\ b}}$ in units of 10^{14} g cm^{-3} during core bounce

Model	*Method N*	*Method R*	*Method A*	*Method CFC*	*Method CFC+*	GR
M5a1	NS	NS	NS	NS	NS	NS
M5c2	O-B	O-A	O-A	O-A → NS	O-A → NS	O-A → NS
M7a4	NS	BH	NS	NS	NS	NS / BH
M7b1	O-B	O-B	O-B	O-B	O-B	O-B
M7c3	O-B	NS	O-A → NS	NS	NS	NS
M8a1	NS	BH	NS	BH	BH	BH
M8c2	O-B	O-B	O-B	O-A	O-A	O-A
M8c4	O-B	NS	NS	NS	NS	NS

Collapse type of the investigated rotating core collapse models

Do we need relativity
for the simulation of core-collapse?

Answer:

In order to have a correct (even qualitatively) description of the core-collapse phenomenon, one needs a relativistic model:

- hydrodynamics (see Pablo's presentations)
- gravity (here)

Einstein's equations

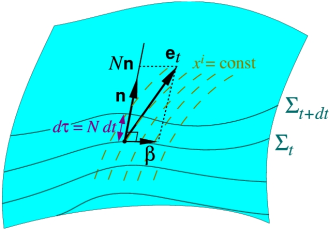
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

3+1 approach and Fully-Constrained Formulation

FCF should soon appear in CoCoNuT…

3+1 FORMALISM

Decomposition of spacetime and of Einstein equations



EVOLUTION EQUATIONS:

$$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}} K_{ij} =$$
$$-D_i D_j N + N R_{ij} - 2N K_{ik} K^k{}_j +$$
$$N \left[K K_{ij} + 4\pi ((S - E)\gamma_{ij} - 2S_{ij}) \right]$$
$$K^{ij} = \frac{1}{2N} \left(\frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i \right).$$

CONSTRAINT EQUATIONS:

$$R + K^2 - K_{ij} K^{ij} = 16\pi E,$$
$$D_j K^{ij} - D^i K = 8\pi J^i.$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -N^2\, dt^2 + \gamma_{ij}\, (dx^i + \beta^i dt)\, (dx^j + \beta^j dt)$$

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Constraint violation

If the constraints are verified for initial data, evolution should preserve them. Therefore, one could in principle solve Einstein equations without solving the constraints

$\Downarrow$

Appearance of constraint violating modes

However, some cures are known :

- solving the constraints at (almost) every time-step …
- using an evolution scheme for which constraint-violating modes remain at a reasonable level (*e.g.* BSSN)
- constraints as evolution equations
- constraint-damping terms and constraint-preserving boundary conditions
- constraint projection
- …

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Some reasons not to solve constraints

Why free evolution schemes are so popular

computational cost of usual elliptic solvers ...

few results of well-posedness for mixed systems versus solid mathematical theory for pure-hyperbolic systems

definition of boundary conditions at finite distance and at black hole excision boundary

Motivations for a fully-constrained scheme

"Alternate" approach (although most straightforward)

- partially constrained schemes: Bardeen & Piran (1983), Stark & Piran (1985), Evans (1986)
- fully constrained schemes: Evans (1989), Shapiro & Teukolsky (1992), Abrahams *et al.* (1994), Choptuik *et al.* (2003), Rinne (2008).

⇒Rather popular for 2D applications, but disregarded in 3D

Still, many advantages:

- constraints are verified!
- elliptic systems have good stability properties
- easy to make link with initial data
- evolution of only two scalar-like fields ...

Usual conformal decomposition

Conformal 3-metric (*e.g.* BSSN:)

$$\tilde{\gamma}_{ij} := \Psi^{-4}\,\gamma_{ij} \text{ or } \gamma_{ij} =: \Psi^{4}\,\tilde{\gamma}_{ij}$$

with

$$\Psi := \left(\frac{\gamma}{f}\right)^{1/12}$$

$$f := \det f_{ij}$$

f_{ij} (with $\dfrac{\partial f_{ij}}{\partial t} = 0$) as the asymptotic structure of γ_{ij}, and $\mathcal{D}_i$ the associated covariant derivative.

Finally,

$$\tilde{\gamma}^{ij} = f^{ij} + h^{ij}$$

is the deviation of the 3-metric from conformal flatness.

$\Rightarrow h^{ij}$ carries the dynamical degrees of freedom of the gravitational field (York, 1972)

Generalized Dirac gauge

Bonazzola *et al.* (2004)

One can generalize the gauge introduced by Dirac (1959) to any type of coordinates:

Divergence-free condition on $\tilde{\gamma}^{ij}$

$$\mathcal{D}_j \tilde{\gamma}^{ij} = \mathcal{D}_j h^{ij} = 0$$

where $\mathcal{D}_j$ denotes the covariant derivative with respect to the flat metric f_{ij}.

Compare

- minimal distortion (Smarr & York 1978) : $D_j \left(\partial \tilde{\gamma}^{ij} / \partial t \right) = 0$
- pseudo-minimal distortion (Nakamura 1994) : $\mathcal{D}^j \left(\partial \tilde{\gamma}_{ij} / \partial t \right) = 0$

Notice: Dirac gauge $\Longleftrightarrow$ BSSN connection functions vanish: $\tilde{\Gamma}^i = 0$

Generalized Dirac gauge properties

- h^{ij} is transverse
- from the requirement $\det \tilde{\gamma}_{ij} = 1$, h^{ij} is asymptotically traceless
- ${}^3R_{ij}$ is a simple Laplacian in terms of h^{ij}
- ${}^3R$ does not contain any second-order derivative of h^{ij}
- with constant mean curvature ($K = t$) and spatial harmonic coordinates ($\mathcal{D}_j \left[(\gamma/f)^{1/2} \gamma^{ij} \right] = 0$), Anderson & Moncrief (2003) have shown that the Cauchy problem is *locally strongly well posed*
- the Conformally-Flat Condition (CFC) verifies the Dirac gauge $\Rightarrow$possibility to easily use many available initial data.

Einstein equations

Dirac gauge and maximal slicing $(K = 0)$

Hamiltonian constraint

$$\Delta(\Psi^2 N) = \Psi^6 N \left(4\pi S + \frac{3}{4}\tilde{A}_{kl}A^{kl}\right) - h^{kl}\mathcal{D}_k\mathcal{D}_l(\Psi^2 N) + \Psi^2 \left[N \left(\frac{1}{16}\tilde{\gamma}^{kl}\mathcal{D}_k h^{ij}\mathcal{D}_l\tilde{\gamma}_{ij} - \frac{1}{8}\tilde{\gamma}^{kl}\mathcal{D}_k h^{ij}\mathcal{D}_j\tilde{\gamma}_{il} + 2\tilde{D}_k \ln\Psi\, \tilde{D}^k \ln\Psi \right) + 2\tilde{D}_k \ln\Psi\, \tilde{D}^k N \right]$$

Momentum constraint

$$\Delta\beta^i + \frac{1}{3}\mathcal{D}^i\left(\mathcal{D}_j\beta^j\right) = 2A^{ij}\mathcal{D}_j N + 16\pi N\Psi^4 J^i - 12 N A^{ij}\mathcal{D}_j \ln\Psi - 2\Delta^i{}_{kl} N A^{kl} - h^{kl}\mathcal{D}_k\mathcal{D}_l\beta^i - \frac{1}{3}h^{ik}\mathcal{D}_k\mathcal{D}_l\beta^l$$

Trace of dynamical equations

$$\Delta N = \Psi^4 N\left[4\pi(E+S) + \tilde{A}_{kl}A^{kl}\right] - h^{kl}\mathcal{D}_k\mathcal{D}_l N - 2\tilde{D}_k \ln\Psi\, \tilde{D}^k N$$

Einstein equations

Dirac gauge and maximal slicing $(K = 0)$

Evolution equations

$$\frac{\partial^2 h^{ij}}{\partial t^2} - \frac{N^2}{\Psi^4}\Delta h^{ij} - 2\pounds_{\boldsymbol{\beta}}\frac{\partial h^{ij}}{\partial t} + \pounds_{\boldsymbol{\beta}}\pounds_{\boldsymbol{\beta}} h^{ij} = \mathcal{S}^{ij}$$

6 components - 3 Dirac gauge conditions - $\left(\det \tilde{\gamma}^{ij} = 1\right)$

2 degrees of freedom

$$-\frac{\partial^2 W}{\partial t^2} + \Delta W = S_W$$

$$-\frac{\partial^2 X}{\partial t^2} + \Delta X = S_X$$

with W and X two scalar potentials related to $h^{\theta\theta} - h^{\varphi\varphi}$ and $h^{\theta\varphi}$.

Conformally-Flat Condition: old and extended formulations

CFC: first version in CoCoNuT

see Dimmelmeier *et al.* (2005)

The CFC reads $h^{ij} = 0$ $\Rightarrow$discarding all gravitational waves! The Einstein system results in 5 coupled non-linear elliptic equations, which sources are with non-compact support:

$$\Delta \ln \Psi = -4\pi\Psi^4 \left(\rho h W^2 - P + \frac{K_{ij}K^{ij}}{16\pi} \right) - \mathcal{D}^i \ln \Psi \, \mathcal{D}_i \ln \Psi,$$

$$\Delta \ln N\Psi = 2\pi\Psi^4 \left(\rho h (3W^2 - 2) + 5P + \frac{7K_{ij}K^{ij}}{16\pi} \right) - \mathcal{D}^i \ln N\Psi \, \mathcal{D}_i \ln N\Psi,$$

$$\Delta \beta^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_k\beta^k = 16\pi N\Psi^4 S^i + 2\Psi^{10}K^{ij}\mathcal{D}_j \left(\frac{N}{\Psi^6} \right) .$$

$\Rightarrow$originally devised by Isenberg (1978), Wilson & Mathews (1989).

Problem with the original formulation

Local uniqueness theorem

Consider the elliptic equation

$$\Delta u + h\, u^p = g \qquad (*)$$

where $p \in \mathbb{R}$ and h and g are independent of u.
If $ph \leq 0$, any solution of $(*)$ is locally unique.

- in the CFC system, this theorem cannot be applied for the equations for Ψ and $N\Psi$;
- During a collapse to a black hole or even during the migration test, the solution of the metric system would jump to a "wrong" one.

This is not due to the CFC approximation! It is happening even in spherical symmetry, where CFC is exact (isotropic gauge)

New (extended) CFC approach

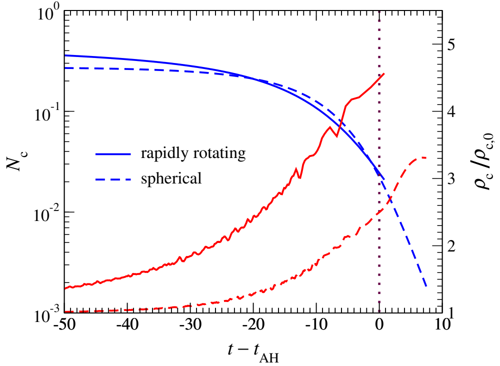
Cordero *et al.* (2008)

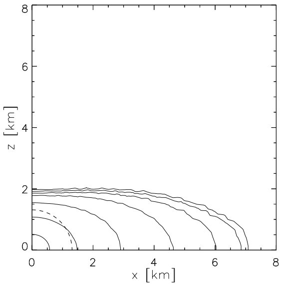
In addition to setting $h^{ij} = 0$, write

$$\hat{A}^{ij} := \Psi^{10} K^{ij} = \mathcal{D}^i X^j + \mathcal{D}^j X^i - \frac{2}{3}\mathcal{D}_k X^k f^{ij} + \cancel{\hat{A}^{ij}_{TT}}$$

- Mom. constraint $\Rightarrow \Delta X^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_j X^j = 8\pi \hat{J}^i$
- Ham. constraint $\Rightarrow \Delta \Psi = -2\pi \frac{\hat{E}}{\Psi} - \frac{f_{il} f_{jm} \hat{A}^{lm} \hat{A}^{ij}}{8\Psi^7}$
- (trace dyn. + Ham. constr.)
  $$\Rightarrow \Delta(N\Psi) = \left[2\pi \Psi^{-2}(\hat{E} + 2\hat{S}) + \frac{7 f_{il} f_{jm} \hat{A}^{lm} \hat{A}^{ij}}{8\Psi^8} \right](N\Psi)$$
- (def. K^{ij} + mom. constr.)
  $$\Rightarrow \Delta \beta^i + \frac{1}{3}\mathcal{D}^i \mathcal{D}_l \beta^l = \frac{N}{\Psi^6}\left(16\pi \hat{J}^i\right) + 2\hat{A}^{ij}\mathcal{D}_j\left(\frac{N}{\Psi^6}\right)$$

Gravitational collapse to a black hole in XCFC





Numerical computation with the XCFC version of CoCoNuT code

Due to the non-uniqueness issue, such a calculation was not possible in CFC, even in spherical symmetry

Rotating relativistic star initial data

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Physical model of rotating neutron stars

Code (available in Lorene) developed for

- self-gravitating perfect fluid in general relativity
- two Killing vector fields (axisymmetry + stationarity)
- Dirac gauge
- equilibrium between matter and gravitational field
- equation of state of a relativistic polytrope $\Gamma = 2$

Considered model here:

- central density $\rho_c = 2.9\rho_{\rm nuc}$
- rotation frequency $f = 641.47$ Hz $\simeq f_{\rm Mass\ shedding}$
- gravitational mass $M_g \simeq 1.51 M_\odot$
- baryon mass $M_b \simeq 1.60 M_\odot$

Equations are the same as in the dynamical case, replacing time derivatives terms by zero

Comparison with ROTSTAR

Lin & Novak (2006)

Other code using quasi-isotropic gauge has been used for a long time and successfully compared to different codes in Nozawa *et al.* (1998).

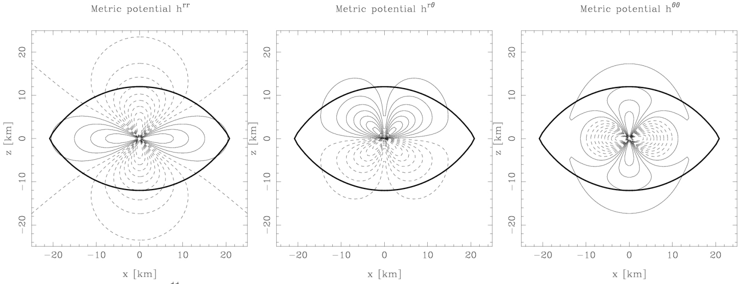
Global quantities

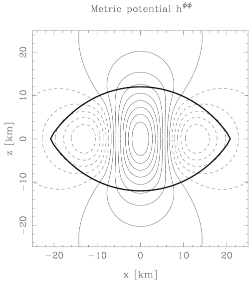
Quantity	q-isotropic	Dirac	rel. diff.
$N(r=0)$	0.727515	0.727522	10^{-5}
$M_g\ [M_\odot]$	1.60142	1.60121	10^{-4}
$M_b\ [M_\odot]$	1.50870	1.50852	10^{-4}
$R_{\rm circ}$ [km]	23.1675	23.1585	4×10^{-4}
$J\ \left[GM_\odot^2/c\right]$	1.61077	1.61032	3×10^{-4}
Virial 2D	1.4×10^{-4}	1.5×10^{-4}	
Virial 3D	2.5×10^{-4}	2.1×10^{-4}	

Virial identities (2 & 3D) are covariant relations that should be fulfilled by any stationary spacetime; they are not imposed numerically.

Stationary axisymmetric models

Deviation from conformal flatness





For all components (except $h^{r\varphi}$ and $h^{\theta\varphi}$, which are null), $h^{ij}_{\max} \sim 0.005$ (up to ~ 0.02 in more compact cases)
$\Rightarrow$comparable with $\gamma_{\theta\theta} - \gamma_{\varphi\varphi}$ in quasi-isotropic gauge

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Trapped surfaces and apparent horizon finder

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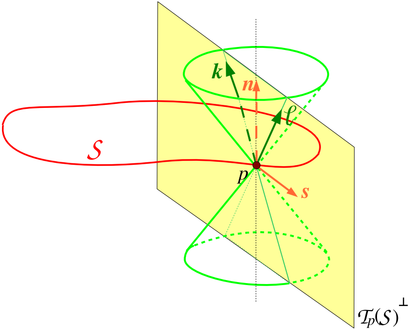
Trapped surfaces

$\mathcal{S}$: closed (i.e. compact without boundary) spacelike 2-dimensional surface embedded in spacetime $(\mathcal{M}, \boldsymbol{g})$



$\exists$ two future-directed null directions (light rays) orthogonal to $\mathcal{S}$:
$\boldsymbol{\ell}$ = outgoing, expansion $\theta^{(\boldsymbol{\ell})}$
$\boldsymbol{k}$ = ingoing, expansion $\theta^{(\boldsymbol{k})}$

In flat space, $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} > 0$

- $\mathcal{S}$ is trapped $\iff$ $\theta^{(\boldsymbol{k})} \leq 0$ and $\theta^{(\boldsymbol{\ell})} \leq 0$
- $\mathcal{S}$ is marginally trapped $\iff$ $\theta^{(\boldsymbol{k})} \leq 0$ and $\theta^{(\boldsymbol{\ell})} = 0$

trapped surface = **local** concept characterizing very strong gravitational fields

Connection with singularities and black holes

Penrose (1965): provided that the weak energy condition holds, $\exists$ a trapped surface $\mathcal{S} \Longrightarrow \exists$ a singularity in $(\mathcal{M}, \boldsymbol{g})$ (in the form of a future inextendible null geodesic)

Hawking & Ellis (1973): provided that the cosmic censorship conjecture holds, $\exists$ a trapped surface $\mathcal{S} \Longrightarrow \exists$ a black hole $\mathcal{B}$ and $\mathcal{S} \subset \mathcal{B}$

$\Rightarrow$local characterization of black holes

AH finder

Lin & Novak (2007)

For any closed smooth 2-surface $\mathcal{S}$ on a time-slice, one thus computes:

- the outward pointing normal unit 3-vector s^i
- the outgoing expansion $\Theta := \theta^{(\boldsymbol{\ell})} = \nabla_i s^i - K + K_{ij} s^i s^j$

An apparent horizon is the outermost marginally trapped surface, therefore the outermost closed 2-surface for which $\Theta = 0$.

Numerically, the AH is defined by $r = h(\theta, \varphi) = \sum_{\ell,m} h_{\ell m} Y_\ell^m(\theta, \varphi)$.

$$\Theta = 0 \iff \Delta_{\theta\varphi} h - 2h = \sigma(h, \gamma_{ij}, K^{ij})$$

which is solved iteratively

$$h_{\ell m} = \frac{-1}{\ell(\ell+1)+2} \int_{\mathcal{S}} Y_\ell^{m*} \sigma d\Omega$$

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