Spectral metric solver and AH finder

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based on collaborations with many people:
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Outline

1. Introduction: metric system
2. Spectral Poisson solvers
3. Interpolation and filtering
4. AH finder
5. Implementation

Einstein equations in CFC

- 3+1 formulation of Einstein equations in terms of (γ_{ij}, K_{ij})
- Conformal Flatness Condition (CFC) : $\gamma_{ij} = \phi^4 f_{ij}$
- no dynamical degree of freedom for the metric — no gravitational waves
- Einstein equations ⇒system of coupled second-order elliptic equations for lapse α, shift β^i and conformal factor ϕ

System of elliptic equations

In the simple version (see Isa's talk, for a more refined one) the metric system writes:

$$\begin{aligned}
\hat{\Delta}\phi &= -2\pi\phi^5\left(\rho h W^2 - P + \frac{K_{ij}K^{ij}}{16\pi}\right), \\
\hat{\Delta}(\alpha\phi) &= 2\pi\alpha\phi^5\left(\rho h(3W^2-2)+5P+\frac{7K_{ij}K^{ij}}{16\pi}\right), \\
\hat{\Delta}\beta^i &= 16\pi\alpha\phi^4 S^i + 2\phi^{10}K^{ij}\hat{\nabla}_j\left(\frac{\alpha}{\phi^6}\right) - \frac{1}{3}\hat{\nabla}^i\hat{\nabla}_k\beta^k,
\end{aligned}$$

with $\hat{\Delta}, \hat{\nabla}$ flat Laplace and gradient operators.

$\Rightarrow$iteration using the solutions of *linear* Poisson equations.

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Spectral methods

- simple viewpoint: representation of a given function using a basis of well know functions
- *e.g.* $\sin, \cos$ (discrete Fourier transform) or orthogonal polynomials (Chebyshev, Legendre, ...)
- if $f(x)$ is a continuous function of $x \in [-1, 1]$, one can approximate it by Chebyshev truncated series

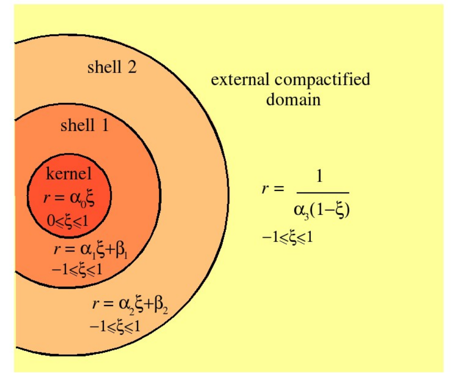
$$f(x) \simeq \sum_{i=0}^{N} c_i T_i(x) \text{ with } T_i(x) = \cos(i \arccos(x)).$$

- f is represented by the vector $\{c_i\}_{i=0\dots N}$ and usual "differential" operators can be seen as matrix multiplications
- $\Rightarrow$solution of ODEs = inversion of $N \times N$ matrices, with N small.

Spherical coordinates

3D Decomposition:

Chebyshev polynomials for ξ,
Y_ℓ^m for the angular part (θ, ϕ)



- symmetries and regularity conditions of the fields at the origin and on the axis of spherical coordinate system
- compactified variable for elliptic PDEs $\Rightarrow$boundary conditions are well imposed

Scalar Poisson solver

The spherical harmonics $Y_\ell^m(\theta,\varphi)$ are eigenvectors of the angular part of the Laplace operator

$$\Delta_{\theta\varphi} Y_\ell^m = -\ell(\ell+1) Y_\ell^m$$

Solution of $\Delta\phi = \sigma$:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right)\phi_{\ell m}(r) = \sigma_{\ell m}(r)$$

Accuracy on the solution $\sim 10^{-13}$ with $N \sim 30$ (exponential decay)

Vector Poisson solver

Vector components expressed in the spherical triad do not behave like scalars: they cannot be expanded onto a basis of $Y_\ell^m(\theta, \varphi)$.

$\Rightarrow$two solutions:

- use Cartesian triad, where $\beta^{x,y,z}(r, \theta, \varphi)$ can be expanded onto Y_ℓ^m, and use the scalar Poisson solver (drawback: needs more points in (θ, φ))
- decompose the spherical components onto pure-spin vector spherical harmonics $(\boldsymbol{Y}^R_{\ell m}, \boldsymbol{Y}^E_{\ell m}, \boldsymbol{Y}^B_{\ell m})$ and solve for the scalar potentials (drawback: more complicated to implement)

$$\begin{aligned} V^r &= \sum_{\ell,m} R_{\ell m}(r) Y_{\ell m}(\theta, \varphi) \\ \eta &= \sum_{\ell,m} E_{\ell m}(r) Y_{\ell m}, \\ \mu &= \sum_{\ell,m} B_{\ell m}(r) Y_{\ell m} \end{aligned}$$

Interpolation between grids

Spectral grid points, from which coefficients are computed, given by the Chebyshev-Gauss-Lobatto rule.
$\Rightarrow$finite-differences (hydro) and spectral (metric) grids do not coincide
$\Rightarrow$interpolation between both spherical grids

- various interpolation algorithms from finite-differences grid to spectral one (piecewise linear or parabolic, splines, minimization of second-derivative) most with $O(N_{\rm spec}^3)$ operations $\Rightarrow$best seems piecewise parabolic
- use of spectral summation (definition of spectral approximation) from spectral grid to finite-differences one: prohibitive cost at $O(N_{FD}^3 \times N_{\rm spec}^3)$ operations
- use of partial summation technique to reduce to $O(N_{FD}^3 \times N_{\rm spec})$ operations

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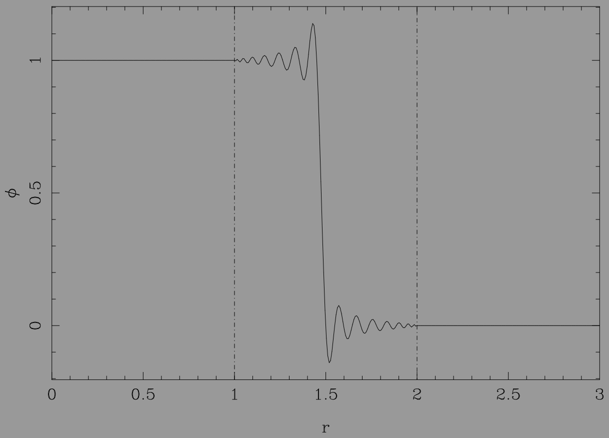
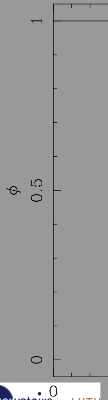
Filtering and Gibbs phenomenon

Discontinuous functions show many spurious oscillations
$\Rightarrow$Gibbs phenomenon

Filtering of coefficients $c_n \mapsto c_n \times e^{-\alpha\left(\frac{n}{N}\right)^{2p}}$

before





AH finder

For any closed smooth 2-surface $\mathcal{S}$ on a time-slice, one can define (see also Pepe's talk)

- the outward pointing normal unit 3-vector s^i
- the expansion $\Theta = \nabla_i s^i - K + K_{ij} s^i s^j$

A marginally trapped surface is definer for $\Theta = 0$

An apparent horizon is the outermost marginally trapped surface.

Numerically, the AH is defined by

$r = h(\theta, \varphi) = \sum_{\ell,m} h_{\ell m} Y_\ell^m(\theta, \varphi)$.

$$\Theta = 0 \iff \Delta_{\theta\varphi} h - 2h = \sigma(h, \gamma_{ij}, K^{ij})$$

which is solved iteratively

$$h_{\ell m} = \frac{-1}{\ell\,(\ell+1)+2} \int_{\mathcal{S}} Y_\ell^{m*} \sigma d\Omega$$

Structure of the code

- evolves the hydro with FORTRAN-HRSC code
- call to the C++ part: interpolation to the spectral grid of hydro sources for metric equations
- iterative solution of the metric system with C++-spectral code
- interpolation back to finite-differences grid of metric potentials

IMPORTANT POINTS:

- most of parameters for the spectral metric solver can be modified at the end of the `parameters` file
- many data and functions related to the spectral metric solver are coded in a class called `Coco` (static variable)
- the domain setup is important for the spectral metric stuff (solver + AH finder): for shells $R_{\rm out}/R_{\rm in} \lesssim 2$
- ... ask me!

References

- H. Dimmelmeier, J. Novak, J.A. Font, J.M. Ibáñez, and E. Müller, Phys. Rev. D**71**(6), 064023, (2005).
- P. Grandclément, S. Bonazzola, E. Gourgoulhon and J.-A. Marck J. Comp. Phys. **170**, 231-260 (2001).
- L.-M. Lin and J. Novak, Class. Quantum Grav. **24**, 2665-2676 (2007)
- J. Novak and J.M. Ibáñez, Astrophys. J. **533** pp.392-405 (2000).