Solving wave equations with spectral methods

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1. The problem to solve
2. Spectral Methods for space / Finite-Differences for time variables
3. Absorbing BC for outgoing waves
4. Numerical results
5. Outlook

Linear Wave Equation of the form:

$$\Box\phi(t,r,\theta,\varphi) = \sigma(t,r,\theta,\varphi)$$

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial r^2} - \frac{2}{r}\frac{\partial\phi}{\partial r} - \frac{1}{r^2}\left(\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial\phi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}\right) = \sigma$$

In General Relativity propagations are usually governed by *non-linear* and *non-flat* wave equations...
$\Rightarrow$ source term σ.

Looking for a *precise* and stable numerical tool to solve the linear wave equation on a *finite* grid.
$\Rightarrow$ absorbing boundary conditions (generalization of Sommerfeld asymptotic radiation condition).

Spatial derivatives are estimated using Spectral Methods, whereas time ones are computed by Finite-Difference schemes

Spectral methods in time have not given good results (except for periodic problems)

Wave equation is decomposed on the basis of $Y_m^l(\theta, \varphi)$

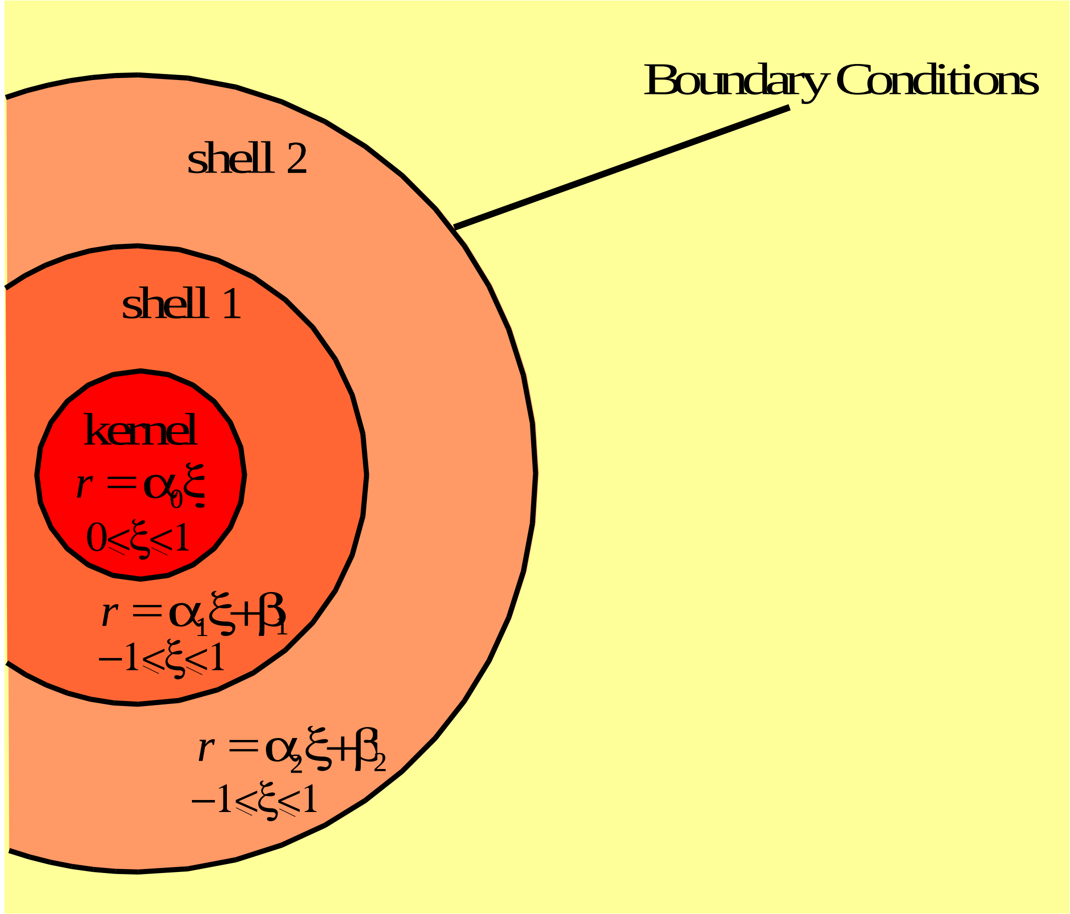
$\Rightarrow$ implicit, second-order time integration is equivalent to an ODE in r:

$$\left[Id - \frac{dt^2}{2}\Delta\right]\phi^{J+1}(r) = 2\phi^J(r) - \phi^{J-1}(r) + \frac{dt^2}{2}\Delta\phi^{J-1}(r) + dt^2\sigma^J(r).$$

which is solved by inverting the matrix of the l.h.s. operator (acting on spectral Chebyshev coefficients).

This is done in each *domain*, a matching being performed so that the solution (and its derivative $/r$) is continuous and verifies Boundary Conditions.

Computational domains



There is no exact BC at a finite distance for outgoing waves

We can:

- either change the formulation of Einstein Equations (CCM, hyperboloïdal formulation, ...)
- or use some "approximate" BC (asymptotic expansion)

Defining

$$L = \frac{\partial}{\partial t} + \frac{\partial}{\partial r},$$

and

$$B_1 = L + \frac{1}{r}, \qquad B_m = \left(L + \frac{2m-1}{r} \right) B_{m-1},$$

we impose

$$B_m \phi|_{r=R} = 0.$$

This is a *multipolar* asymptotic expansion: $B_m \phi = 0$ means all modes up to $l = m - 1$ are let out.

Gravitational waves are at least quadrupolar

So we have tried: $B_3\phi = 0$, which can be explicited:

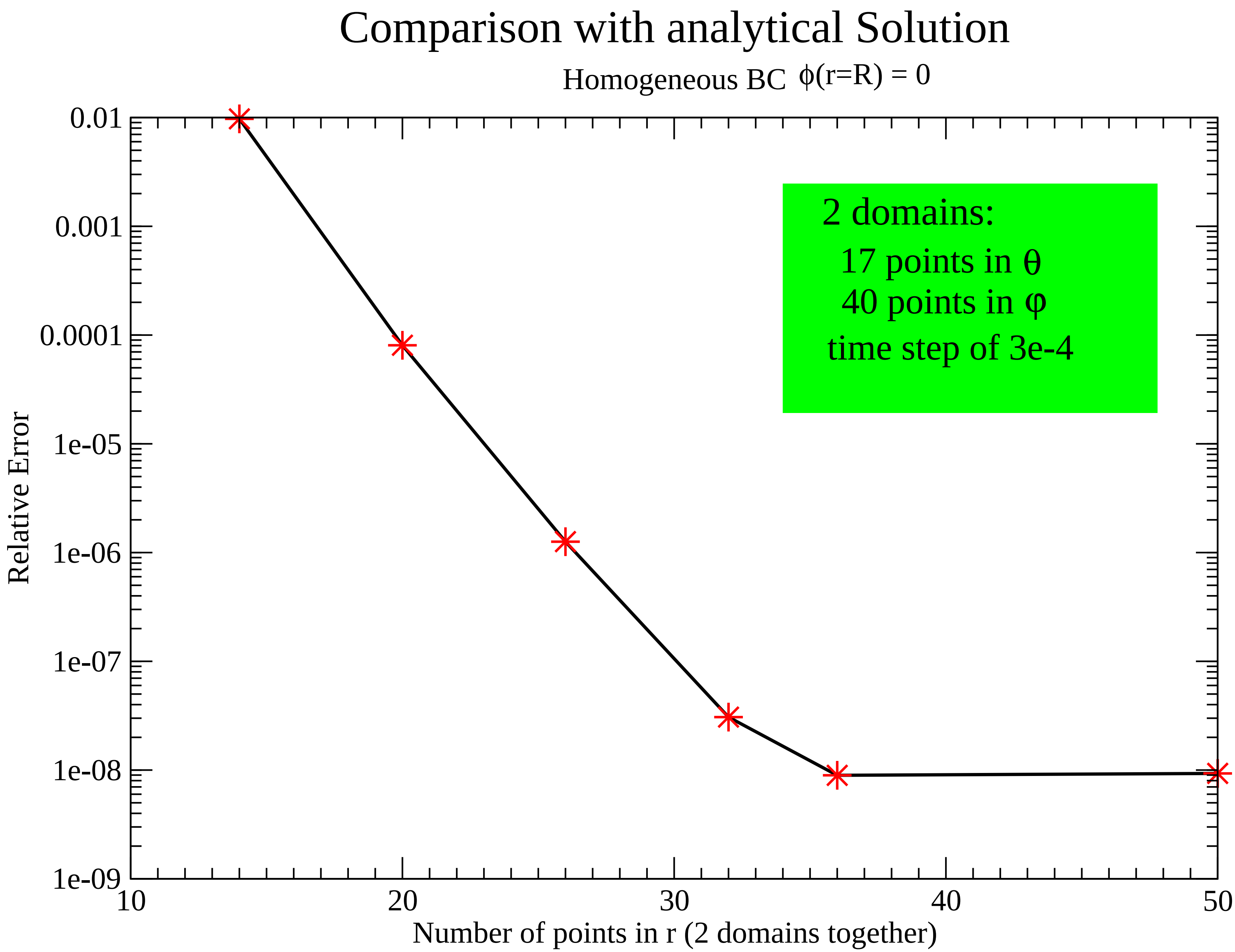
$$\left(\frac{\partial^3}{\partial t^3} + 3\frac{\partial^3}{\partial t^2 \partial r} + 9\frac{1}{r}\frac{\partial^2}{\partial t^2} + 3\frac{\partial^3}{\partial t \partial r^2} + 18\frac{1}{r^2}\frac{\partial}{\partial t} + 18\frac{1}{r}\frac{\partial^2}{\partial t \partial r} + \frac{\partial^3}{\partial r^3} + 9\frac{1}{r}\frac{\partial^2}{\partial r^2} + 18\frac{1}{r^2}\frac{\partial}{\partial r} + 6\frac{1}{r^3}\right)\phi\bigg|_{r=R} = 0.$$

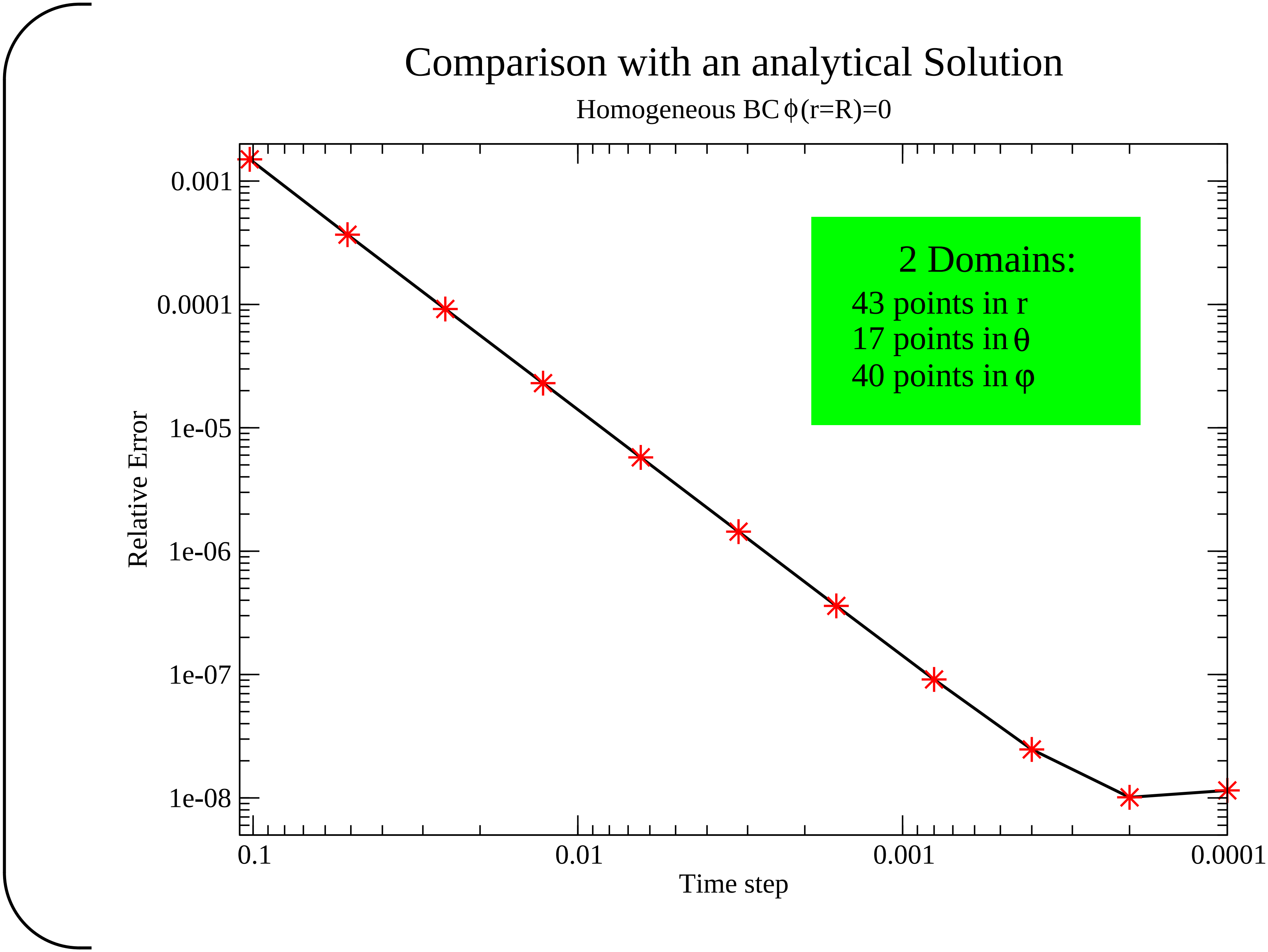
Using the fact that $\Box\phi = 0$, one gets:

$$\forall(\theta,\varphi), \quad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{r}\right)\phi(\theta,\varphi)\bigg|_{r=R} = \xi_1(\theta,\varphi), \qquad \text{with}$$

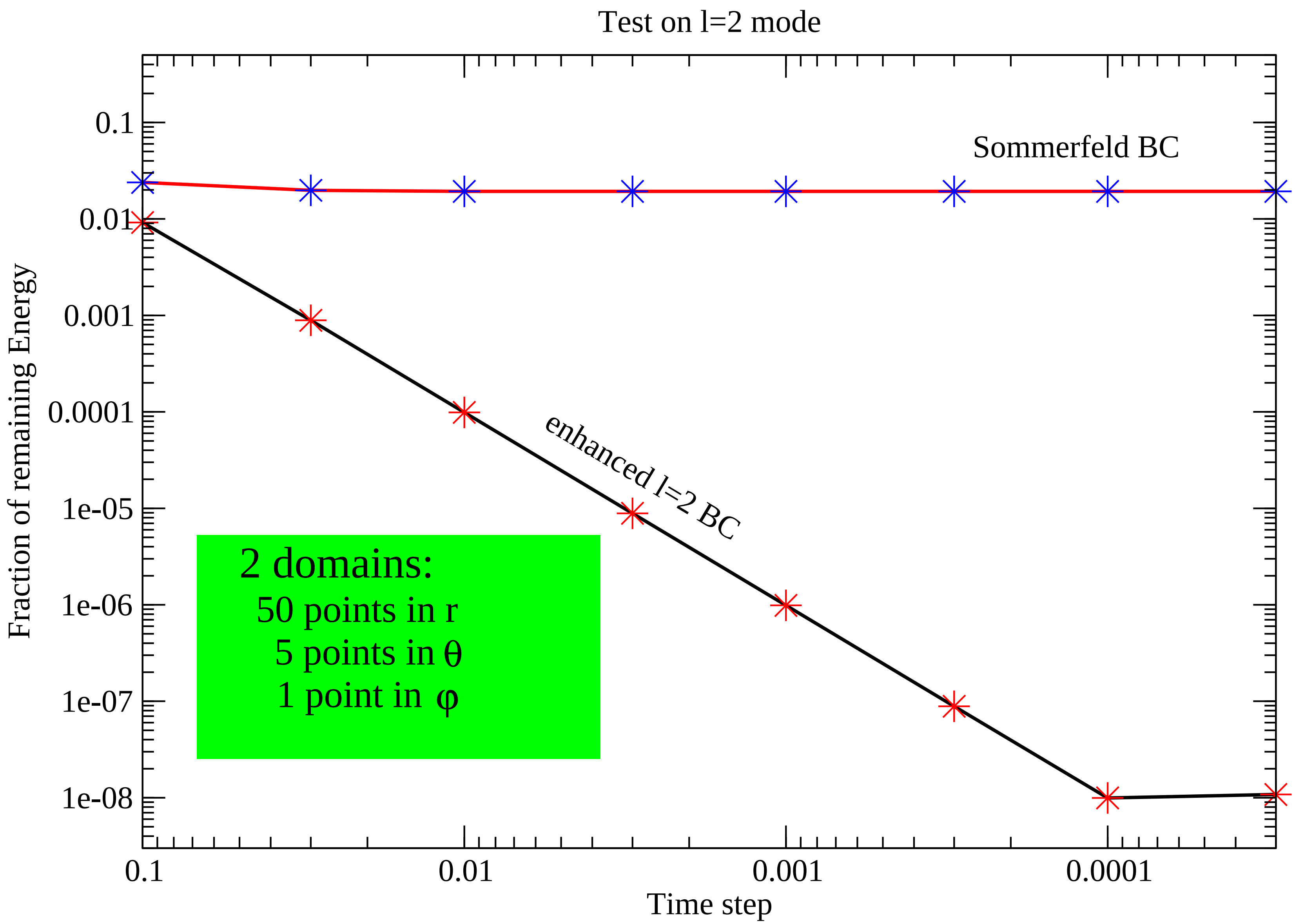
$$\frac{\partial^2 \xi_1}{\partial t^2} - \frac{3}{4R^2}\Delta_{\mathsf{ang}}\xi_1 + \frac{3}{R}\frac{\partial \xi_1}{\partial t} + \frac{3\xi_1}{2R^2} = \frac{1}{2R^2}\Delta_{\mathsf{ang}}\left(\frac{\phi}{R} - \frac{\partial \phi}{\partial r}\bigg|_{r=R}\right).$$

This *wave equation on a sphere* being very easily integrated when decomposed on spherical harmonics ($\Delta_{\mathsf{ang}} Y_l^m = -l(l+1)Y_l^m$).

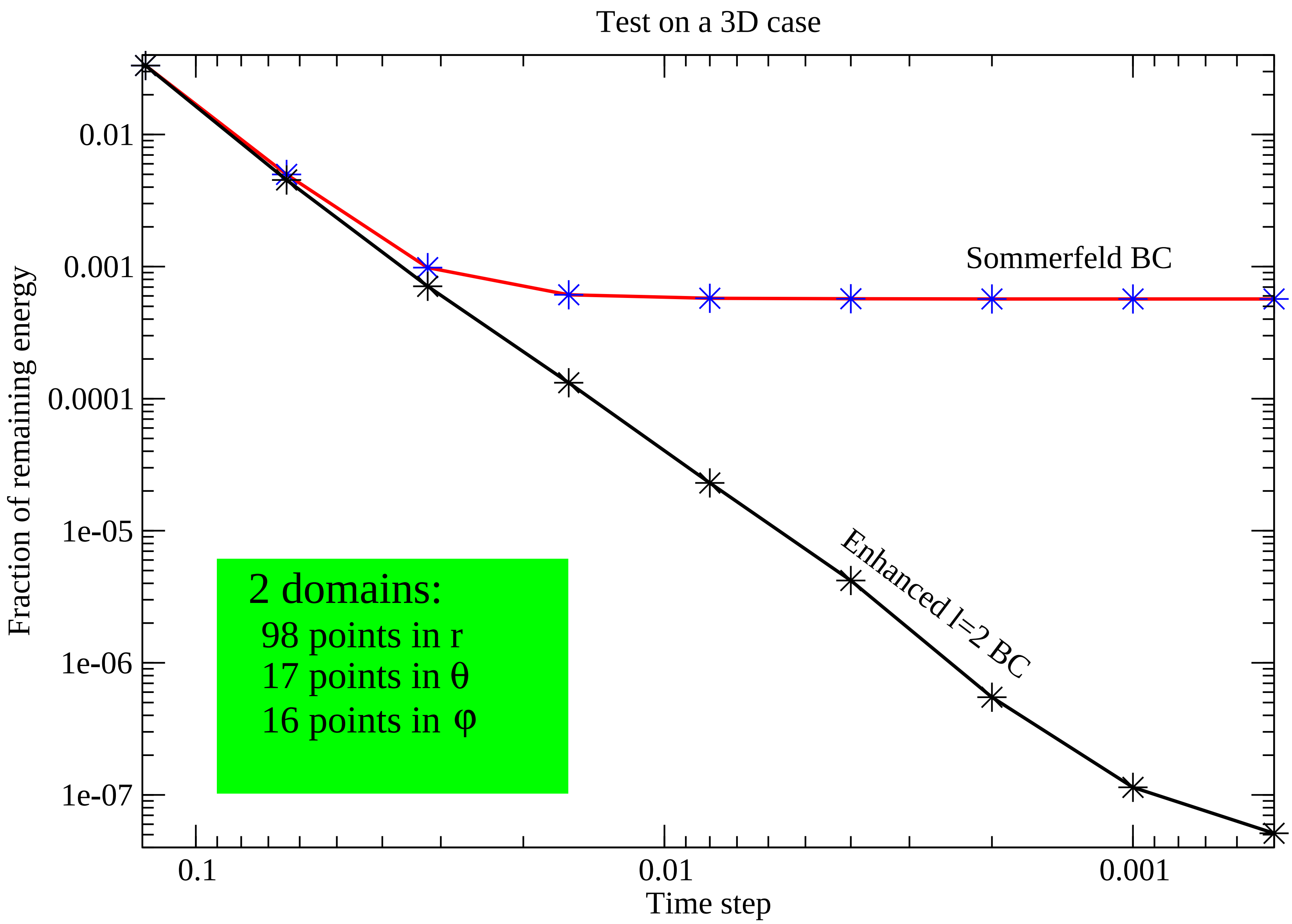

Comparison with analytical Solution
Homogeneous BC φ(r=R) = 0
2 domains:
17 points in θ
40 points in φ
time step of 3e-4
Relative Error
0.01
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
1e-09
10
20
30
40
50
Number of points in r (2 domains together)


Comparison with an analytical Solution
Homogeneous BC ϕ(r=R)=0
2 Domains:
43 points in r
17 points in θ
40 points in φ
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
Relative Error
0.1
0.01
0.001
0.0001
Time step

Comparison between Sommerfeld and enhanced BCs



Comparison between Sommerfeld and enhanced BCs



Outlook

- implement and test higher order ($B_5 \rightarrow \xi_2$,...),
- develop *physical* BCs: e.g. post-Minkowskian approach,
- compare with characteristic-Cauchy matching,
- try spectral decomposition in time!