

A FULLY-CONSTRAINED FORMULATION OF EINSTEIN EQUATIONS: SETUP AND NUMERICAL IMPLEMENTATION

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PLAN

1 INTRODUCTION

- Constraints Issues in 3+1 Formalism

2 DESCRIPTION OF THE FORMULATION AND STRATEGY

- Covariant 3+1 Conformal Decomposition
- Einstein Equations and Integration Strategy
- Boundary Conditions and Black Holes

3 NUMERICAL METHODS

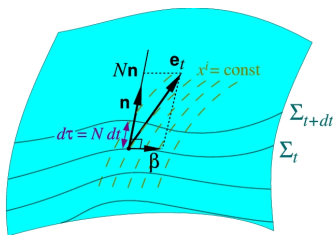
- Multidomain Spectral Methods with Spherical Coordinates
- Solutions of Poisson and Wave Equations
- Spherical Coordinates and Tensor Components

4 METHODS FOR DIVERGENCE-FREE EVOLUTIONS

- Pure-spin Spherical Harmonics
- New System for Time Evolution
- Method for Tensors

3+1 FORMALISM

Decomposition of spacetime and of Einstein equations



EVOLUTION EQUATIONS:

$$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_\beta K_{ij} =$$

$$-D_i D_j N + N R_{ij} - 2N K_{ik} K_j^k +$$

$$N [K K_{ij} + 4\pi((S - E)\gamma_{ij} - 2S_{ij})]$$

$$K^{ij} = \frac{1}{2N} \left(\frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i \right).$$

CONSTRAINT EQUATIONS:

$$R + K^2 - K_{ij} K^{ij} = 16\pi E,$$

$$D_j K^{ij} - D^i K = 8\pi J^i.$$

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt) (dx^j + \beta^j dt)$$

FREE VS. CONSTRAINED FORMULATIONS

As in electromagnetism, if the constraints are satisfied initially, they remain so for a solution of the evolution equations.

FREE EVOLUTION

- start with initial data verifying the constraints,
- solve **only** the 6 evolution equations,
- recover a solution of **all** Einstein equations.

⇒ apparition of **constraint violating modes** from round-off errors. Considered cures:

- Using of constraint damping terms and adapted gauges (many groups).
- Solving the constraints at every time-step (efficient elliptic solver?).

Description of Formulation and Strategy

Bonazzola et al. (2004)

USUAL CONFORMAL DECOMPOSITION

Standard definition of conformal 3-metric (e.g.
Baumgarte-Shapiro-Shibata-Nakamura – BSSN formalism)

DYNAMICAL DEGREES OF FREEDOM OF THE GRAVITATIONAL FIELD:

York (1972) : they are carried by the conformal “metric”

$$\hat{\gamma}_{ij} := \gamma^{-1/3} \gamma_{ij} \quad \text{with } \gamma := \det \gamma_{ij}$$

PROBLEM

$\hat{\gamma}_{ij}$ = *tensor density* of weight $-2/3$
not always easy to deal with tensor densities... not *really*
covariant!

INTRODUCTION OF A FLAT METRIC

We introduce f_{ij} (with $\frac{\partial f_{ij}}{\partial t} = 0$) as the asymptotic structure of γ_{ij} , and \mathcal{D}_i the associated covariant derivative.

DEFINE:

$$\begin{aligned}\tilde{\gamma}_{ij} &:= \Psi^{-4} \gamma_{ij} \text{ or } \gamma_{ij} =: \Psi^4 \tilde{\gamma}_{ij} \\ &\text{with} \\ \Psi &:= \left(\frac{\gamma}{f}\right)^{1/12} \\ f &:= \det f_{ij}\end{aligned}$$

$\tilde{\gamma}_{ij}$ is invariant under any conformal transformation of γ_{ij} and verifies $\det \tilde{\gamma}_{ij} = f$

\Rightarrow no more tensor densities: only tensors.

Finally,

$$\tilde{\gamma}^{ij} = f^{ij} + h^{ij}$$

is the deviation of the 3-metric from conformal flatness.

CONFORMAL FLATNESS CONDITION

Within conformal 3+1 formalism, one imposes that $h^{ij} = 0$:

$$\gamma_{ij} = \psi^4 f_{ij}$$

with f_{ij} the flat metric and $\psi(t, x^1, x^2, x^3)$ the conformal factor. First devised by Isenberg in 1978 as a **waveless approximation** to GR, it has been widely used for generating initial data, ...

SET OF 5 NON-LINEAR ELLIPTIC PDES ($K = 0$)

$$\Delta\psi = -2\pi\psi^{-1} \left(E^* + \frac{\psi^6 K_{ij} K^{ij}}{16\pi} \right),$$

$$\Delta(N\psi) = 2\pi N\psi^{-1} \left(E^* + 2S^* + \frac{7\psi^6 K_{ij} K^{ij}}{16\pi} \right),$$

$$\Delta\beta^i + \frac{1}{3}\nabla^i\nabla_j\beta^j = 16\pi N\psi^{-2}(S^*)^i + 2\psi^{10}K^{ij}\nabla_j\frac{N}{\psi^6}.$$

GENERALIZED DIRAC GAUGE

One can generalize the gauge introduced by Dirac (1959) to any type of coordinates:

DIVERGENCE-FREE CONDITION ON $\tilde{\gamma}^{ij}$

$$\mathcal{D}_j \tilde{\gamma}^{ij} = \mathcal{D}_j h^{ij} = 0$$

where \mathcal{D}_j denotes the covariant derivative with respect to the flat metric f_{ij} .

Compare

- minimal distortion (Smarr & York 1978) : $\mathcal{D}_j (\partial \tilde{\gamma}^{ij} / \partial t) = 0$
- pseudo-minimal distortion (Nakamura 1994) :
 $\mathcal{D}^j (\partial \tilde{\gamma}_{ij} / \partial t) = 0$

Notice: Dirac gauge \iff BSSN connection functions vanish:
 $\tilde{\Gamma}^i = 0$

GENERALIZED DIRAC GAUGE PROPERTIES

- h^{ij} is transverse
- from the requirement $\det \tilde{\gamma}_{ij} = 1$, h^{ij} is asymptotically traceless
- ${}^3R_{ij}$ is a simple Laplacian in terms of h^{ij}
- 3R does not contain any second-order derivative of h^{ij}
- with constant mean curvature ($K = t$) and spatial harmonic coordinates ($\mathcal{D}_j \left[(\gamma/f)^{1/2} \gamma^{ij} \right] = 0$), Anderson & Moncrief (2003) have shown that the Cauchy problem is *locally strongly well posed*
- the **Conformal Flat Condition (CFC)** verifies the Dirac gauge \Rightarrow possibility to easily use initial data for binaries now available

EINSTEIN EQUATIONS

DIRAC GAUGE AND MAXIMAL SLICING ($K = 0$)

HAMILTONIAN CONSTRAINT

$$\Delta\Psi = -2\pi E\Psi^5 - \frac{\Psi^5}{8}\tilde{A}_{kl}A^{kl} - h^{kl}\mathcal{D}_k\mathcal{D}_l\Psi + \frac{\Psi}{8}\tilde{R}$$

MOMENTUM CONSTRAINT

$$\begin{aligned}\Delta\beta^i + \frac{1}{3}\mathcal{D}^i(\mathcal{D}_j\beta^j) &= 2A^{ij}\mathcal{D}_jN + 16\pi N\Psi^4J^i - 12NA^{ij}\mathcal{D}_j\ln\Psi - 2\Delta^i{}_{kl}NA^{kl} \\ &\quad - h^{kl}\mathcal{D}_k\mathcal{D}_l\beta^i - \frac{1}{3}h^{ik}\mathcal{D}_k\mathcal{D}_l\beta^l\end{aligned}$$

TRACE OF DYNAMICAL EQUATIONS

$$\Delta N = \Psi^4N \left[4\pi(E + S) + \tilde{A}_{kl}A^{kl} \right] - h^{kl}\mathcal{D}_k\mathcal{D}_lN - 2\tilde{D}_k\ln\Psi\tilde{D}^kN$$

EINSTEIN EQUATIONS

DIRAC GAUGE AND MAXIMAL SLICING ($K = 0$)

EVOLUTION EQUATIONS

$$\frac{\partial^2 h^{ij}}{\partial t^2} - \frac{N^2}{\Psi^4} \Delta h^{ij} - 2\mathcal{L}_\beta \frac{\partial h^{ij}}{\partial t} + \mathcal{L}_\beta \mathcal{L}_\beta h^{ij} = \mathcal{S}^{ij}$$

6 components - 3 Dirac gauge conditions - ($\det \tilde{\gamma}^{ij} = 1$)

2 DEGREES OF FREEDOM

$$\begin{aligned} -\frac{\partial^2 A}{\partial t^2} + \Delta A &= S_A \\ -\frac{\partial^2 \tilde{B}}{\partial t^2} + \Delta \tilde{B} &= S_{\tilde{B}} \end{aligned}$$

with A and \tilde{B} two scalar potentials representing the degrees of freedom.

INTEGRATION PROCEDURE

Everything is known on slice Σ_t



Evolution of A and \tilde{B} to next time-slice Σ_{t+dt} (+ hydro)



Deduce $h_{\text{Traceless}}^{ij}(t+dt)$ from Dirac and trace-free conditions



Deduce the trace from $\det \tilde{\gamma}^{ij} = 1$; thus $h^{ij}(t+dt)$ and $\tilde{\gamma}^{ij}(t+dt)$.



Iterate on the system of elliptic equations for N , $\Psi^2 N$ and β^i on Σ_{t+dt}

OUTGOING BOUNDARY CONDITIONS

- If no compactification is done, it is necessary to impose boundary condition at a finite distance R ;
- Far enough from the source, one can consider the evolution operator as being a flat D'Alembert operator;
- It is then possible to use outgoing-wave boundary condition.

BUT

- Usual outgoing-wave condition (Sommerfeld) is exact, up to numerical scheme precision, only for $\ell = 0$ mode.

⇒ Use of enhanced condition (Novak & Bonazzola (2004)):

- exact (up to discretization error) $\forall \ell \leq 2$,
- for $\ell > 2$, the reflected wave decreases as $1/R^4$ (versus $1/R^2$ for Sommerfeld).

BOUNDARY CONDITIONS AT A BLACK HOLE HORIZON

UNDER DEVELOPMENT...

- Use of excision technique for black hole evolution \Rightarrow at the *apparent horizon* (Gourgoulhon & Jaramillo (2006));
- In this region, the evolution operator for h^{ij} must be taken with all (linear) terms,

Then, in the Dirac gauge, for a **dynamical horizon**:

- All characteristics are outgoing...
- ... no boundary condition must be imposed
(Cordero-Carrión et al. (2008))

\Rightarrow OK with the intuition of a spacelike boundary of the computational domain.

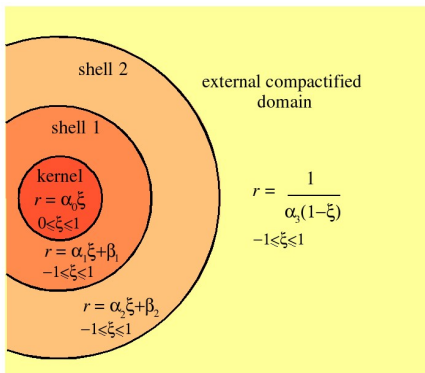
In the stationary case, first numerical solution imposing only from boundary conditions, in fully-constrained scheme by Vasset et al. (2009).

Numerical Methods

Grandclément & Novak (2009)

MULTIDOMAIN 3D DECOMPOSITION

NUMERICAL LIBRARY LORENE
(<http://www.lorene.obspm.fr>)



DECOMPOSITION:

Chebyshev polynomials for ξ ,
Fourier or Y_ℓ^m for the
angular part (θ, ϕ) ,

- symmetries and regularity conditions of the fields at the origin and on the axis of spherical coordinate system
- compactified variable for elliptic PDEs
 \Rightarrow boundary conditions are well imposed

SOLUTIONS OF POISSON AND WAVE EQUATIONS

The angular part of any field ϕ is decomposed on a set of spherical harmonics $Y_\ell^m(\theta, \varphi)$, which are eigenvectors of the angular part of the Laplace operator

$$\Delta_{\theta\varphi} Y_\ell^m = -\ell(\ell+1) Y_\ell^m$$

$$\Delta\phi = \sigma$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} \right) \phi_{\ell m}(r) = \sigma_{\ell m}(r)$$

Accuracy on the solution
 $\sim 10^{-13}$ (exponential decay)

$$\square\phi = \sigma$$

$$\left[1 - \frac{\delta t^2}{2} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} \right) \right] \phi_{\ell m}^{J+1} = \sigma_{\ell m}^J$$

Accuracy on the solution
 $\sim 10^{-10}$ (time-differencing)

$\forall(\ell, m)$ the operator inversion \iff inversion of a $\sim 30 \times 30$ matrix

Non-linear parts are evaluated in the physical space and contribute as sources to the equations.

SPHERICAL COORDINATES AND COMPONENTS

CHOICE FOR f_{ij} : SPHERICAL POLAR COORDINATES

- stars and black holes are of spheroidal shape
- compactification made easy (only r)
- use of spherical harmonics
- grid boundaries are smooth surfaces

USE OF SPHERICAL ORTHONORMAL TRIAD (TENSOR COMPONENTS)

- Dirac gauge can easily be imposed
- asymptotically, it is easier to extract gravitational waves

Methods for divergence-free Evolutions

Novak et al. (2010)

OBJECTIVE:

SOLVE THE TENSOR WAVE EQUATION UNDER DIVERGENCE-FREE CONSTRAINTS

$$\begin{aligned}\forall t \geq 0, \forall r < R, & \quad \frac{\partial^2 h^{ij}}{\partial t^2} = \Delta h^{ij}, \\ \forall t \geq 0, \forall r \leq R, & \quad \nabla_j h^{ij} = 0, \\ \forall r \leq R, & \quad h^{ij}(0, r, \theta, \varphi) = \alpha_0^{ij}(r, \theta, \varphi), \\ \forall r \leq R, & \quad \left. \frac{\partial h^{ij}}{\partial t} \right|_{t=0} = \gamma_0^{ij}(r, \theta, \varphi), \\ \forall t \geq 0, & \quad h^{ij}(t, R, \theta, \varphi) = \beta_0^{ij}(t, \theta, \varphi).\end{aligned}$$

⇒ First, consider the **vector case** (easier!).

VECTOR SPHERICAL HARMONICS

FOLLOWING *e.g.* THORNE (1980)

A 3D vector field \mathbf{V} can be decomposed onto a set of **vector spherical harmonics**

$$\mathbf{V} = \sum_{\ell,m} R_{\ell m}(r) \mathbf{Y}_{\ell m}^R(\theta, \varphi) + E_{\ell m}(r) \mathbf{Y}_{\ell m}^E(\theta, \varphi) + B_{\ell m}(r) \mathbf{Y}_{\ell m}^B(\theta, \varphi),$$

- **pure spin** vector harmonics,
- orthonormal set of regular angular functions,
- not eigenfunctions of vector angular Laplacian

$$\mathbf{Y}_{\ell m}^R \propto Y_{\ell m} \mathbf{r}, \text{ (longitudinal)}$$

$$\mathbf{Y}_{\ell m}^E \propto \mathcal{D}Y_{\ell m}, \text{ (transverse)}$$

$$\mathbf{Y}_{\ell m}^B \propto \mathbf{r} \times \mathcal{D}Y_{\ell m} \text{ (transverse)}$$

$V^r = \sum_{\ell,m} R_{\ell m}(r) Y_{\ell m}(\theta, \varphi)$, and we define two other potentials

$$V^\theta = \frac{\partial \eta}{\partial \theta} - \frac{1}{\sin \theta} \frac{\partial \mu}{\partial \varphi},$$

$$V^\varphi = \frac{1}{\sin \theta} \frac{\partial \eta}{\partial \varphi} + \frac{\partial \mu}{\partial \theta};$$

$$\eta(r, \theta, \varphi) = \sum_{\ell,m} E_{\ell m}(r) Y_{\ell m},$$

$$\mu(r, \theta, \varphi) = \sum_{\ell,m} B_{\ell m}(r) Y_{\ell m}$$

NEW EQUATIONS

FLAT WAVE OPERATOR $\square V^i = S^i$ (DIVERGENCE-FREE CASE)

$$\begin{aligned} -\frac{\partial^2 V^r}{\partial t^2} + \Delta V^r + \frac{2}{r} \frac{\partial V^r}{\partial r} + \frac{2V^r}{r^2} &= S^r, \\ -\frac{\partial^2 \eta}{\partial t^2} + \Delta \eta + \frac{2}{r} \frac{\partial V^r}{\partial r} &= \eta_S, \\ -\frac{\partial^2 \mu}{\partial t^2} + \Delta \mu &= \mu_S. \end{aligned}$$

DIVERGENCE-FREE CONDITION $\mathcal{D}_i V^i = 0$

$$\frac{\partial V^r}{\partial r} + \frac{2V^r}{r} + \frac{1}{r} \Delta_{\theta\varphi} \eta = 0$$

... thus μ does not depend on the divergence of \mathbf{V} .

HELMHOLTZ DECOMPOSITION

Any vector field \mathbf{V} on \mathbb{R}^3 , twice continuously differentiable and with rapid enough decay at infinity can be uniquely written as

$$\mathbf{V} = \tilde{\mathbf{V}} + \mathcal{D}\phi, \text{ with } \mathcal{D}_i \tilde{V}^i = 0.$$

from $\mathcal{D} \times \mathbf{V} = \mathcal{D} \times \tilde{\mathbf{V}}$, one gets

$$\begin{aligned} \mu_V &= \mu_{\tilde{V}} \text{ (twice: } r\text{- and } \eta\text{- components) ,} \\ \frac{\partial \eta_V}{\partial r} + \frac{\eta_V}{r} - \frac{V^r}{r} &= \frac{\partial \eta_{\tilde{V}}}{\partial r} + \frac{\eta_{\tilde{V}}}{r} - \frac{\tilde{V}^r}{r} \text{ (}\mu\text{- component) .} \end{aligned}$$

\Rightarrow the quantities

$$A = \frac{\partial \eta}{\partial r} + \frac{\eta}{r} - \frac{V^r}{r}$$

and μ are not sensitive to the gradient part of a vector.

EVOLUTION EQUATIONS

ENSURING DIVERGENCE-FREE CONDITION...

From the definition of A and the expression of the wave operator for a vector, one gets for the source ($\square V^i = S^i$)

$$A_S = \frac{\partial \eta_S}{\partial r} + \frac{\eta_S}{r} - \frac{S^r}{r},$$

and

$$\square A_V = A_S$$

once A is known, one can reconstruct the vector V^i from

$$\begin{aligned} \frac{\partial \eta}{\partial r} + \frac{\eta}{r} - \frac{V^r}{r} &= A, \\ \frac{\partial V^r}{\partial r} + \frac{2V^r}{r} + \frac{1}{r} \Delta_{\theta\varphi} \eta &= 0 \text{ divergence-free condition.} \end{aligned}$$

and μ (since $\square \mu = \mu_S$).

INTEGRATION PROCEDURE

- ① from S^i compute A_S and μ_S ,
- ② solve the equation for μ ,
- ③ solve the equation for A ,
- ④ solve the coupled system given by the divergence-free condition and the definition of A to get V^r and η ,
- ⑤ reconstruct V^i from V^r , η and μ .

TENSOR SPHERICAL HARMONICS

A 3D symmetric tensor field \mathbf{h} can be decomposed onto a set of **tensor pure spin spherical harmonics** and one can get 6 scalar potentials to represent the tensor:

\mathbf{T}^{L_0}	\mathbf{T}^{T_0}	\mathbf{T}^{E_1}	\mathbf{T}^{B_1}	\mathbf{T}^{E_2}	\mathbf{T}^{B_2}
h^{rr}	$\tau = h^{\theta\theta} + h^{\varphi\varphi}$	η	μ	W	X

with the following relations:

$$h^{r\theta} = \frac{\partial \eta}{\partial \theta} - \frac{1}{\sin \theta} \frac{\partial \mu}{\partial \varphi},$$

$$h^{r\varphi} = \frac{1}{\sin \theta} \frac{\partial \eta}{\partial \varphi} + \frac{\partial \mu}{\partial \theta},$$

$$\frac{h^{\theta\theta} - h^{\varphi\varphi}}{2} = \frac{\partial^2 W}{\partial \theta^2} - \frac{1}{\tan \theta} \frac{\partial W}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2 W}{\partial \varphi^2} - 2 \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial X}{\partial \varphi} \right),$$

$$h^{\theta\varphi} = \frac{\partial^2 X}{\partial \theta^2} - \frac{1}{\tan \theta} \frac{\partial X}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2 X}{\partial \varphi^2} + 2 \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial W}{\partial \varphi} \right).$$

DIFFERENTIAL OPERATORS

DIVERGENCE-FREE CONDITION $H^i = \mathcal{D}_j h^{ij} = 0$

$$H^r = \frac{\partial h^{rr}}{\partial r} + \frac{2h^{rr}}{r} + \frac{1}{r} \Delta_{\theta\varphi} \eta - \frac{\tau}{r} = 0,$$

$$H^\eta = \frac{\partial \eta}{\partial r} + \frac{3\eta}{r} + (\Delta_{\theta\varphi} + 2) \frac{W}{r} + \frac{\tau}{2r} = 0,$$

$$H^\mu = \frac{\partial \mu}{\partial r} + \frac{3\mu}{r} + (\Delta_{\theta\varphi} + 2) X = 0;$$

“ELECTRIC TYPE” POTENTIALS

$$h^{rr}, \tau, \eta, W$$

“MAGNETIC TYPE”

$$\mu, X$$

⇒ two groups of coupled equations for the wave operator.

DIVERGENCE-FREE PART OF A SYMMETRIC TENSOR

As for the Helmholtz decomposition:

$$h^{ij} = \tilde{h}^{ij} + \mathcal{D}^i V^j + \mathcal{D}^j V^i$$

... but no possibility to use the curl operator on a symmetric tensor!

3 DEGREES OF FREEDOM FOR \tilde{h}

$$A = \frac{\partial X}{\partial r} - \frac{\mu}{r},$$

$$B = \frac{\partial W}{\partial r} - \frac{1}{2r} \Delta_{\theta\varphi} W - \frac{\eta}{r} + \frac{\tau}{4r},$$

$$C = \frac{\partial \tau}{\partial r} - \frac{2h^{rr}}{r} - 2\Delta_{\theta\varphi} \left(\frac{\partial W}{\partial r} + \frac{W}{r} \right).$$

WAVE EQUATION

$$\square h^{ij} = S^{ij}$$

$$\square A = A_S,$$

$$\square B + \frac{C}{2r^2} = B_S,$$

$$\square C - \frac{2C}{r^2} - \frac{8\Delta_{\theta\varphi} B}{r^2} = C_S.$$

DIVERGENCE-FREE EVOLUTION

DEFINE ℓ BY ℓ

$$\tilde{B}_{\ell m} = 2B_{\ell m} + \frac{C_{\ell m}}{2(\ell+1)},$$

$$\tilde{C}_{\ell m} = 2B_{\ell m} - \frac{C_{\ell m}}{2\ell};$$

WAVE EQUATION $\square h^{ij} = S^{ij}$

$$\square \tilde{B} + \frac{2\ell \tilde{B}}{r^2} = \tilde{B}_S,$$

$$\square \tilde{C} - \frac{2(\ell+1)\tilde{C}}{r^2} = \tilde{C}_S.$$

In the case where $f_{ij}h^{ij} = 0$ ($h^{rr} = -\tau$):

- 1 compute A_S and \tilde{B}_S ,
- 2 solve wave equations for A and \tilde{B} (a wave operator shifted in ℓ),
- 3 solve the system composed of
 - definition of A
 - $H^\mu = 0$ (Dirac gauge)
 - definition of \tilde{B}
 - $H^r = 0$
 - $H^\eta = 0$

on the one hand, and

- 4 recover the tensor components.

on the other hand,

SUMMARY - PERSPECTIVES

- A fully-constrained formalism of Einstein equations, aimed at obtaining stable solutions in astrophysical scenarios (with matter) has been presented, implemented and tested ;
- This formalism has been implemented in a numerical library using spectral methods with spherical coordinates and spherical tensor components;
- A method, based on this library, has been devised to solve the evolution equations and ensure the gauge at spectral accuracy.

Future directions:

- Implementation of the newer version of the FCF (avoiding uniqueness problems) and tests in the case of gravitational wave collapse;
- Use of the CFC approach together with excision methods in the collapse code to simulate the formation of a black hole (work by N. Vasset);