A fully-constrained formulation of Einstein equations: setup and numerical implementation

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based on collaboration with
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Plan

1. Introduction
2. Description of the Formulation and Strategy
3. Numerical Methods
4. Methods for Divergence-free Evolutions

Plan

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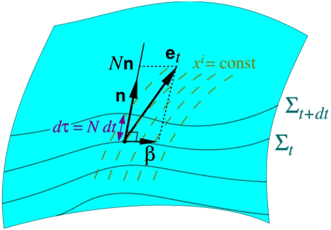
3+1 FORMALISM

Decomposition of spacetime and of Einstein equations



EVOLUTION EQUATIONS:

$$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_\beta K_{ij} =$$
$$-D_i D_j N + N R_{ij} - 2N K_{ik} K^k_{\ j} +$$
$$N\left[K K_{ij} + 4\pi((S-E)\gamma_{ij} - 2S_{ij})\right]$$
$$K^{ij} = \frac{1}{2N}\left(\frac{\partial \gamma^{ij}}{\partial t} + D^i\beta^j + D^j\beta^i\right).$$

CONSTRAINT EQUATIONS:

$$R + K^2 - K_{ij}K^{ij} = 16\pi E,$$
$$D_j K^{ij} - D^i K = 8\pi J^i.$$

$$g_{\mu\nu}\,dx^\mu\,dx^\nu = -N^2\,dt^2 + \gamma_{ij}\,(dx^i + \beta^i dt)\,(dx^j + \beta^j dt)$$

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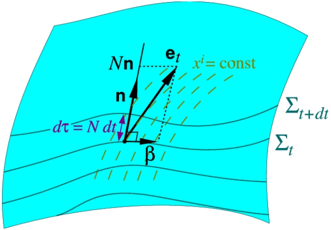

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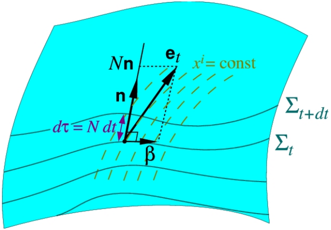

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Free vs. constrained formulations

As in electromagnetism, if the constraints are satisfied initially, they remain so for a solution of the evolution equations.

Free evolution

- start with initial data verifying the constraints,
- solve only the 6 evolution equations,
- recover a solution of all Einstein equations.

$\Rightarrow$apparition of constraint violating modes from round-off errors. Considered cures:

- Using of constraint damping terms and adapted gauges (many groups).
- Solving the constraints at every time-step (efficient elliptic solver?).

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Description of Formulation and Strategy

Bonazzola et al. (2004)

Usual conformal decomposition

Standard definition of conformal 3-metric (e.g. Baumgarte-Shapiro-Shibata-Nakamura – BSSN formalism)

Dynamical degrees of freedom of the gravitational field:

York (1972) : they are carried by the conformal "metric"

$$\hat{\gamma}_{ij} := \gamma^{-1/3}\, \gamma_{ij} \qquad \text{with } \gamma := \det \gamma_{ij}$$

Problem

$\hat{\gamma}_{ij}$ = *tensor density* of weight $-2/3$
not always easy to deal with tensor densities... not *really* covariant!

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Introduction of a flat metric

We introduce f_{ij} (with $\dfrac{\partial f_{ij}}{\partial t} = 0$) as the asymptotic structure of γ_{ij}, and $\mathcal{D}_i$ the associated covariant derivative.

Define:

$$\tilde{\gamma}_{ij} := \Psi^{-4}\gamma_{ij} \text{ or } \gamma_{ij} =: \Psi^{4}\tilde{\gamma}_{ij}$$

with

$$\Psi := \left(\frac{\gamma}{f}\right)^{1/12}$$

$$f := \det f_{ij}$$

$\tilde{\gamma}_{ij}$ is invariant under any conformal transformation of γ_{ij} and verifies $\det \tilde{\gamma}_{ij} = f$

$\Rightarrow$ *no more tensor densities: only tensors.*

Finally,

$$\tilde{\gamma}^{ij} = f^{ij} + h^{ij}$$

is the deviation of the 3-metric from conformal flatness.

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Conformal flatness condition

Within conformal 3+1 formalism, one imposes that $h^{ij} = 0$:

$$\gamma_{ij} = \psi^4 f_{ij}$$

with f_{ij} the flat metric and $\psi(t, x^1, x^2, x^3)$ the conformal factor. First devised by Isenberg in 1978 as a waveless approximation to GR, it has been widely used for generating initial data, ...

Set of 5 non-linear elliptic PDEs ($K = 0$)

$$\Delta\psi = -2\pi\psi^{-1}\left(E^* + \frac{\psi^6 K_{ij}K^{ij}}{16\pi}\right),$$

$$\Delta(N\psi) = 2\pi N\psi^{-1}\left(E^* + 2S^* + \frac{7\psi^6 K_{ij}K^{ij}}{16\pi}\right),$$

$$\Delta\beta^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_j\beta^j = 16\pi N\psi^{-2}(S^*)^i + 2\psi^{10}K^{ij}\mathcal{D}_j\frac{N}{\psi^6}.$$

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Generalized Dirac gauge

One can generalize the gauge introduced by Dirac (1959) to any type of coordinates:

Divergence-free condition on $\tilde{\gamma}^{ij}$

$$\mathcal{D}_j \tilde{\gamma}^{ij} = \mathcal{D}_j h^{ij} = 0$$

where $\mathcal{D}_j$ denotes the covariant derivative with respect to the flat metric f_{ij}.

Compare

- minimal distortion (Smarr & York 1978) : $D_j \left(\partial \tilde{\gamma}^{ij} / \partial t \right) = 0$
- pseudo-minimal distortion (Nakamura 1994) : $\mathcal{D}^j \left(\partial \tilde{\gamma}_{ij} / \partial t \right) = 0$

Notice: Dirac gauge $\Longleftrightarrow$ BSSN connection functions vanish: $\tilde{\Gamma}^i = 0$

Generalized Dirac gauge properties

- h^{ij} is transverse
- from the requirement $\det \tilde{\gamma}_{ij} = 1$, h^{ij} is asymptotically traceless
- ${}^3R_{ij}$ is a simple Laplacian in terms of h^{ij}
- ${}^3R$ does not contain any second-order derivative of h^{ij}
- with constant mean curvature ($K = t$) and spatial harmonic coordinates ($\mathcal{D}_j \left[(\gamma/f)^{1/2} \gamma^{ij} \right] = 0$), Anderson & Moncrief (2003) have shown that the Cauchy problem is *locally strongly well posed*
- the Conformal Flat Condition (CFC) verifies the Dirac gauge ⇒possibility to easily use initial data for binaries now available

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Einstein equations

Dirac gauge and maximal slicing ($K = 0$)

Hamiltonian constraint

$$\Delta\Psi = -2\pi E\Psi^5 - \frac{\Psi^5}{8}\tilde{A}_{kl}A^{kl} - h^{kl}\mathcal{D}_k\mathcal{D}_l\Psi + \frac{\Psi}{8}\tilde{R}$$

Momentum constraint

$$\Delta\beta^i + \frac{1}{3}\mathcal{D}^i\left(\mathcal{D}_j\beta^j\right) = 2A^{ij}\mathcal{D}_jN + 16\pi N\Psi^4J^i - 12NA^{ij}\mathcal{D}_j\ln\Psi - 2\Delta^i{}_{kl}NA^{kl}$$
$$-h^{kl}\mathcal{D}_k\mathcal{D}_l\beta^i - \frac{1}{3}h^{ik}\mathcal{D}_k\mathcal{D}_l\beta^l$$

Trace of dynamical equations

$$\Delta N = \Psi^4N\left[4\pi(E+S) + \tilde{A}_{kl}A^{kl}\right] - h^{kl}\mathcal{D}_k\mathcal{D}_lN - 2\tilde{D}_k\ln\Psi\,\tilde{D}^kN$$

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$$\frac{\partial^2 h^{ij}}{\partial t^2} - \frac{N^2}{\Psi^4}\Delta h^{ij} - 2\pounds_{\boldsymbol{\beta}}\frac{\partial h^{ij}}{\partial t} + \pounds_{\boldsymbol{\beta}}\pounds_{\boldsymbol{\beta}} h^{ij} = \mathcal{S}^{ij}$$

6 components - 3 Dirac gauge conditions - ($\det \tilde{\gamma}^{ij} = 1$)

2 degrees of freedom

$$-\frac{\partial^2 A}{\partial t^2} + \Delta A = S_A$$

$$-\frac{\partial^2 \tilde{B}}{\partial t^2} + \Delta \tilde{B} = S_{\tilde{B}}$$

with A and $\tilde{B}$ two scalar potentials representing the degrees of freedom.

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Integration procedure

Everything is know on slice Σ_t

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Evolution of A and $\tilde{B}$ to next time-slice Σ_{t+dt} (+ hydro)

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Deduce $h^{ij}_{\text{Traceless}}(t+dt)$ from Dirac and trace-free conditions

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Deduce the trace from $\det \tilde{\gamma}^{ij} = 1$; thus $h^{ij}(t+dt)$ and $\tilde{\gamma}^{ij}(t+dt)$.

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Iterate on the system of elliptic equations for N, $\Psi^2 N$ and β^i on Σ_{t+dt}

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Outgoing boundary conditions

- If no compactification is done, it is necessary to impose boundary condition at a finite distance R;
- Far enough from the source, one can consider the evolution operator as being a flat Dalembert operator;
- It is then possible to use outgoing-wave boundary condition.

BUT

- Usual outgoing-wave condition (Sommerfeld) is exact, up to numerical scheme precision, only for $\ell = 0$ mode.

$\Rightarrow$Use of enhanced condition (Novak & Bonazzola (2004)):

- exact (up to discretization error) $\forall \ell \leq 2$,
- for $\ell > 2$, the reflected wave decreases as $1/R^4$ (versus $1/R^2$ for Sommerfeld).

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Boundary conditions at a black hole horizon

Under development...

- Use of excision technique for black hole evolution ⇒at the *apparent horizon* (*Gourgoulhon & Jaramillo (2006)*);
- In this region, the evolution operator for h^{ij} must be taken with all (linear) terms,

Then, in the Dirac gauge, for a dynamical horizon:

- All characteristics are outgoing...
- ... no boundary condition must be imposed (*Cordero-Carrión et al. (2008)*)

⇒OK with the intuition of a spacelike boundary of the computational domain.

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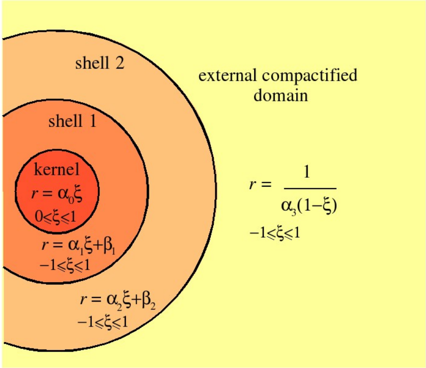
Numerical Methods

Grandclément & Novak (2009)

Multidomain 3D decomposition

numerical library Lorene
(http://www.lorene.obspm.fr)



Decomposition:

Chebyshev polynomials for ξ, Fourier or Y_ℓ^m for the angular part (θ, ϕ),

- symmetries and regularity conditions of the fields at the origin and on the axis of spherical coordinate system
- compactified variable for elliptic PDEs $\Rightarrow$boundary conditions are well imposed

Solutions of Poisson and wave equations

The angular part of any field ϕ is decomposed on a set of spherical harmonics $Y_\ell^m(\theta,\varphi)$, which are eigenvectors of the angular part of the Laplace operator

$$\Delta_{\theta\varphi} Y_\ell^m = -\ell(\ell+1) Y_\ell^m$$

$\Delta\phi = \sigma$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right)\phi_{\ell m}(r) = \sigma_{\ell m}(r)$$

Accuracy on the solution $\sim 10^{-13}$ (exponential decay)

$\Box\phi = \sigma$

$$\left[1 - \frac{\delta t^2}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right)\right]\phi_{\ell m}^{J+1} = \sigma_{\ell m}^{J}$$

Accuracy on the solution $\sim 10^{-10}$ (time-differencing)

$\forall(\ell,m)$ the operator inversion $\Longleftrightarrow$ inversion of a $\sim 30\times 30$ matrix

Non-linear parts are evaluated in the physical space and contribute as sources to the equations.

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$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right)\phi_{\ell m}(r) = \sigma_{\ell m}(r)$$

Accuracy on the solution $\sim 10^{-13}$ (exponential decay)

$\Box\phi = \sigma$

$$\left[1 - \frac{\delta t^2}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right)\right]\phi_{\ell m}^{J+1} = \sigma_{\ell m}^{J}$$

Accuracy on the solution $\sim 10^{-10}$ (time-differencing)

$\forall(\ell, m)$ the operator inversion $\Longleftrightarrow$ inversion of a $\sim 30\times 30$ matrix

Non-linear parts are evaluated in the physical space and contribute as sources to the equations.

Solutions of Poisson and wave equations

The angular part of any field ϕ is decomposed on a set of spherical harmonics $Y_\ell^m(\theta, \varphi)$, which are eigenvectors of the angular part of the Laplace operator

$$\Delta_{\theta\varphi} Y_\ell^m = -\ell(\ell+1) Y_\ell^m$$

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Spherical coordinates and components

Choice for f_{ij} : spherical polar coordinates

- stars and black holes are of spheroidal shape
- compactification made easy (only r)
- use of spherical harmonics
- grid boundaries are smooth surfaces

Use of spherical orthonormal triad (tensor components)

- Dirac gauge can easily be imposed
- asymptotically, it is easier to extract gravitational waves

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Methods for divergence-free Evolutions

Novak et al. (2010)

Objective:

Solve the tensor wave equation under divergence-free constraints

$$
\begin{aligned}
\forall t \geq 0,\ \forall r < R, &\qquad \frac{\partial^2 h^{ij}}{\partial t^2} = \Delta h^{ij},\\
\forall t \geq 0,\ \forall r \leq R, &\qquad \mathcal{D}_j h^{ij} = 0,\\
\forall r \leq R, &\qquad h^{ij}(0, r, \theta, \varphi) = \alpha_0^{ij}(r, \theta, \varphi),\\
\forall r \leq R, &\qquad \left.\frac{\partial h^{ij}}{\partial t}\right|_{t=0} = \gamma_0^{ij}(r, \theta, \varphi),\\
\forall t \geq 0, &\qquad h^{ij}(t, R, \theta, \varphi) = \beta_0^{ij}(t, \theta, \varphi).
\end{aligned}
$$

⇒First, consider the vector case (easier!).

Vector spherical harmonics

Following *e.g.* Thorne (1980)

A 3D vector field $\boldsymbol{V}$ can be decomposed onto a set of vector spherical harmonics

$$\boldsymbol{V} = \sum_{\ell,m} R_{\ell m}(r)\boldsymbol{Y}^R_{\ell m}(\theta,\varphi) + E_{\ell m}(r)\boldsymbol{Y}^E_{\ell m}(\theta,\varphi) + B_{\ell m}(r)\boldsymbol{Y}^B_{\ell m}(\theta,\varphi),$$

- pure spin vector harmonics,
- orthonormal set of regular angular functions,
- not eigenfunctions of vector angular Laplacian

$$\begin{aligned}
\boldsymbol{Y}^R_{\ell m} &\propto Y_{\ell m}\boldsymbol{r}, \text{ (longitudinal)}\\
\boldsymbol{Y}^E_{\ell m} &\propto \boldsymbol{\mathcal{D}}Y_{\ell m}, \text{ (transverse)}\\
\boldsymbol{Y}^B_{\ell m} &\propto \boldsymbol{r}\times\boldsymbol{\mathcal{D}}Y_{\ell m} \text{ (transverse)}
\end{aligned}$$

$V^r = \sum R_{\ell m}(r)Y_{\ell m}(\theta,\varphi)$, and we define two other potentials

$$\begin{aligned}
V^\theta &= \frac{\partial\eta}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial\mu}{\partial\varphi},\\
V^\varphi &= \frac{1}{\sin\theta}\frac{\partial\eta}{\partial\varphi} + \frac{\partial\mu}{\partial\theta};
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\eta(r,\theta,\varphi) &= \sum_{\ell,m} E_{\ell m}(r)Y_{\ell m},\\
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Flat wave operator $\Box V^i = S^i$ (divergence-free case)

$$
\begin{aligned}
-\frac{\partial^2 V^r}{\partial t^2} + \Delta V^r + \frac{2}{r}\frac{\partial V^r}{\partial r} + \frac{2V^r}{r^2} &= S^r, \\
-\frac{\partial^2 \eta}{\partial t^2} + \Delta \eta + \frac{2}{r}\frac{\partial V^r}{\partial r} &= \eta_S, \\
-\frac{\partial^2 \mu}{\partial t^2} + \Delta \mu &= \mu_S.
\end{aligned}
$$

Divergence-free condition $\mathcal{D}_i V^i = 0$

$$
\frac{\partial V^r}{\partial r} + \frac{2V^r}{r} + \frac{1}{r}\Delta_{\theta\varphi}\eta = 0
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... thus μ does not depend on the divergence of V.

New equations

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Helmholtz decomposition

Any vector field $\boldsymbol{V}$ on $\mathbb{R}^3$, twice continuously differentiable and with rapid enough decay at infinity can be uniquely written as

$$\boldsymbol{V} = \tilde{\boldsymbol{V}} + \boldsymbol{\mathcal{D}}\phi, \text{ with } \mathcal{D}_i\tilde{V}^i = 0.$$

from $\boldsymbol{\mathcal{D}} \times \boldsymbol{V} = \boldsymbol{\mathcal{D}} \times \tilde{\boldsymbol{V}}$, one gets

$$\mu_V = \mu_{\tilde{V}} \text{ (twice: } r\text{- and } \eta\text{- components)},$$

$$\frac{\partial \eta_V}{\partial r} + \frac{\eta_V}{r} - \frac{V^r}{r} = \frac{\partial \eta_{\tilde{V}}}{\partial r} + \frac{\eta_{\tilde{V}}}{r} - \frac{\tilde{V}^r}{r} \text{ (}\mu\text{- component)}.$$

⇒the quantities

$$A = \frac{\partial \eta}{\partial r} + \frac{\eta}{r} - \frac{V^r}{r}$$

and μ are not sensitive to the gradient part of a vector.

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Evolution equations

ensuring divergence-free condition...

From the definition of A and the expression of the wave operator for a vector, one gets for the source ($\square V^i = S^i$)

$$A_S = \frac{\partial \eta_S}{\partial r} + \frac{\eta_S}{r} - \frac{S^r}{r},$$

and

$$\square A_V = A_S$$

once A is known, one can reconstruct the vector V^i from

$$\frac{\partial \eta}{\partial r} + \frac{\eta}{r} - \frac{V^r}{r} = A,$$

$$\frac{\partial V^r}{\partial r} + \frac{2V^r}{r} + \frac{1}{r}\Delta_{\theta\varphi}\eta = 0 \text{ divergence-free condition.}$$

and μ (since $\square \mu = \mu_S$).

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Integration procedure
(vector case)

1. from S^i compute A_S and μ_S,
2. advance in time μ, solving its wave equation,
3. advance in time A, solving its equation,
4. solve the coupled system given by the divergence-free condition and the definition of A to get the new V^r and η,
5. reconstruct V^i at new times-step from V^r, η and μ.

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Laboratoire Univers et Théories

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Tensor spherical harmonics

A 3D symmetric tensor field $\boldsymbol{h}$ can be decomposed onto a set of tensor pure spin spherical harmonics and one can get 6 scalar potentials to represent the tensor:

$\boldsymbol{T}^{L_0}$	$\boldsymbol{T}^{T_0}$	$\boldsymbol{T}^{E_1}$	$\boldsymbol{T}^{B_1}$	$\boldsymbol{T}^{E_2}$	$\boldsymbol{T}^{B_2}$
h^{rr}	$\tau = h^{\theta\theta} + h^{\varphi\varphi}$	η	μ	W	X

with the following relations:

$$h^{r\theta} = \frac{\partial \eta}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial \mu}{\partial \varphi},$$

$$h^{r\varphi} = \frac{1}{\sin\theta}\frac{\partial \eta}{\partial \varphi} + \frac{\partial \mu}{\partial \theta},$$

$$\frac{h^{\theta\theta} - h^{\varphi\varphi}}{2} = \frac{\partial^2 W}{\partial \theta^2} - \frac{1}{\tan\theta}\frac{\partial W}{\partial \theta} - \frac{1}{\sin^2\theta}\frac{\partial^2 W}{\partial \varphi^2} - 2\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial X}{\partial \varphi}\right),$$

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\frac{h^{\theta\theta} - h^{\varphi\varphi}}{2} &= \frac{\partial^2 W}{\partial \theta^2} - \frac{1}{\tan\theta}\frac{\partial W}{\partial \theta} - \frac{1}{\sin^2\theta}\frac{\partial^2 W}{\partial \varphi^2} - 2\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial X}{\partial \varphi}\right), \\
h^{\theta\varphi} &= \frac{\partial^2 X}{\partial \theta^2} - \frac{1}{\tan\theta}\frac{\partial X}{\partial \theta} - \frac{1}{\sin^2\theta}\frac{\partial^2 X}{\partial \varphi^2} + 2\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial W}{\partial \varphi}\right).
\end{aligned}
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Differential operators

Divergence-free condition $H^i = \mathcal{D}_j h^{ij} = 0$

$$
\begin{aligned}
H^r &= \frac{\partial h^{rr}}{\partial r} + \frac{2h^{rr}}{r} + \frac{1}{r}\Delta_{\theta\varphi}\eta - \frac{\tau}{r} = 0, \\
H^\eta &= \frac{\partial \eta}{\partial r} + \frac{3\eta}{r} + \left(\Delta_{\theta\varphi} + 2\right)\frac{W}{r} + \frac{\tau}{2r} = 0, \\
H^\mu &= \frac{\partial \mu}{\partial r} + \frac{3\mu}{r} + \left(\Delta_{\theta\varphi} + 2\right) X = 0;
\end{aligned}
$$

"electric type" potentials

h^{rr}, τ, η, W

"magnetic type"

μ, X

$\Rightarrow$two groups of coupled equations for the wave operator.

Divergence-free part of a symmetric tensor

As for the Helmholtz decomposition:

$$h^{ij} = \tilde{h}^{ij} + \mathcal{D}^i V^j + \mathcal{D}^j V^i$$

... but no possibility to use the curl operator on a symmetric tensor!

3 degrees of freedom for $\tilde{h}$

$$\begin{aligned} A &= \frac{\partial X}{\partial r} - \frac{\mu}{r}, \\ B &= \frac{\partial W}{\partial r} - \frac{1}{2r}\Delta_{\theta\varphi} W - \frac{\eta}{r} + \frac{\tau}{4r}, \\ C &= \frac{\partial \tau}{\partial r} - \frac{2h^{rr}}{r} - 2\Delta_{\theta\varphi}\left(\frac{\partial W}{\partial r} + \frac{W}{r}\right). \end{aligned}$$

Wave equation $\Box h^{ij} = S^{ij}$

$$\Box A = A_S,$$

$$\Box B + \frac{C}{2r^2} = B_S,$$

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Divergence-free evolution

Define ℓ by ℓ

$$\tilde{B}_{\ell m} = 2B_{\ell m} + \frac{C_{\ell m}}{2(\ell+1)},$$

$$\tilde{C}_{\ell m} = 2B_{\ell m} - \frac{C_{\ell m}}{2\ell};$$

Wave equation $\Box h^{ij} = S^{ij}$

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In the case where $f_{ij}h^{ij} = 0$ ($h^{rr} = -\tau$):

1. compute A_S and $\tilde{B}_S$,
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   for (μ, X) on the one hand, and
   - definition of $\tilde{B}$
   - $H^r = 0$
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   for (h^{rr}, τ, η, W) on the other hand,
4. recover the tensor components.

Divergence-free evolution

Define ℓ by ℓ

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- A fully-constrained formalism of Einstein equations, aimed at obtaining stable solutions in astrophysical scenarios (with matter) has been presented, implemented and tested ;
- This formalism has been implemented in the numerical library LORENE using spectral methods with spherical coordinates and spherical tensor components;
- A method, based on this library, has been devised to solve the evolution equations and ensure the gauge at spectral accuracy.

Future directions:

- Implementation of the newer version of the FCF (avoiding uniqueness problems) and tests in the case of gravitational wave collapse;
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