Constrained evolution of gravitational-wave spacetimes

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18th International Conference on General Relativity and Gravitation, Sydney, July 13th

Outline

Choice of formulation of Einstein equations

10 degrees of freedom: e.g. in 3+1 formalism N, β^i, γ^{ij}.
4 constraint equations + 6 evolution equations - 4 gauge conditions

Free Evolution

- constraints are not solved during the evolution;
- they are solved initially and supposed to be evolved by the evolution equations;
- lapse and shift are prescribed as gauge degrees of freedom;
- the 3-metric is advanced in time by evolution equations.

Constrained Evolution

- the constraints are solved at every time-step, giving the shift and the determinant of γ^{ij};
- the lapse is obtained from the slicing gauge choice through an elliptic equation;
- only 2 evolution equations are solved (two polarizations of the gravitational field);
- the rest of γ^{ij} is obtained through the 3 remaining gauge conditions and the determinant.

Flat metric and Dirac gauge

Following Bonazzola *et al.* (2004)

Conformal 3+1 (a.k.a BSSN) formulation, but use of f_{ij} (with $\frac{\partial f_{ij}}{\partial t} = 0$) as the asymptotic structure of γ_{ij}, and $\mathcal{D}_i$ the associated covariant derivative.

Conformal factor Ψ

$$\tilde{\gamma}_{ij} := \Psi^{-4}\gamma_{ij} \text{ with } \Psi := \left(\frac{\gamma}{f}\right)^{1/12}, \text{ so } \det\tilde{\gamma}_{ij} = f$$

Finally,

$$\tilde{\gamma}^{ij} = f^{ij} + h^{ij}$$

is the deviation of the 3-metric from conformal flatness.

Generalization the gauge introduced by Dirac (1959) to any type of coordinates:

divergence-free condition on $\tilde{\gamma}^{ij}$

$$\mathcal{D}_j\tilde{\gamma}^{ij} = \mathcal{D}_j h^{ij} = 0$$

+ Maximal slicing ($K = 0$)

Einstein equations

Dirac gauge and maximal slicing

Constraint Equations

$$\begin{aligned} \Delta\Psi &= \mathcal{S}_{\mathrm{Ham}}, \\ \Delta\beta^i + \frac{1}{3}\mathcal{D}^i\left(\mathcal{D}_j\beta^j\right) &= \mathcal{S}_{\mathrm{Mom}}. \end{aligned}$$

Trace of dynamical equations

$$\Delta N = \mathcal{S}_{\dot{K}}$$

Dynamical equations

$$\frac{\partial^2 h^{ij}}{\partial t^2} - \frac{N^2}{\Psi^4}\Delta h^{ij} - 2\pounds_{\boldsymbol{\beta}}\frac{\partial h^{ij}}{\partial t} + \pounds_{\boldsymbol{\beta}}\pounds_{\boldsymbol{\beta}} h^{ij} = \mathcal{S}^{ij}_{\mathrm{Dyn}}$$

$\Rightarrow$overdetermined system

Two algorithms to solve the tensor wave equation

h^{ij} decomposed onto tensor spherical harmonics (Zerilli, 1970) $\Rightarrow$definition of potentials $h^{rr}, \eta, \mu, X, W, \tau$.

Deriving (following Bonazzola *et al.* 2004)

$$\Box h^{ij} = \sigma^{ij} \Rightarrow \begin{cases} \Box(r^2 h^{rr}) = r^2 \sigma^{rr} \\ \Box \mu = \mu(\sigma^{ij}) \end{cases}$$

- advantage: the other potentials are easily deduced through the gauge conditions deriving h^{rr} and μ
- drawback: many radial derivatives $\Rightarrow$too much numerical noise

Integrating

h^{ij} is divergence-free $\Rightarrow$only three scalar degrees of freedom $A, \tilde{B}, h$ easily expressed from $h^{rr}, \eta, \mu, X, W, \tau$.

$$\Box h^{ij} = \sigma^{ij} \Rightarrow \begin{cases} \Box A = A(\sigma^{ij}) \\ \tilde{\Box} \tilde{B} = \tilde{B}(\sigma^{ij}) \end{cases}$$

- drawback: need to solve elliptic system from gauge conditions
- advantage: much more robust and accurate

Regularity and boundary conditions

Use of spherical grid and coordinates, with multi-domain spectral methods.

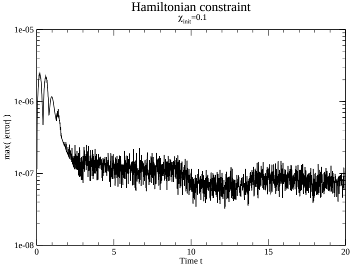
- h^{ij} is represented through 4 scalar and 2 pseudo-scalar potentials
- standard regularity conditions on the z-axis taken care by numerical decomposition on scalar spherical harmonics $Y_\ell^m(\theta, \varphi)$
- standard regularity conditions at the origin $f_{\ell,m}(r) \sim r^\ell$ approximatively fulfilled by even/odd Chebyshev decomposition

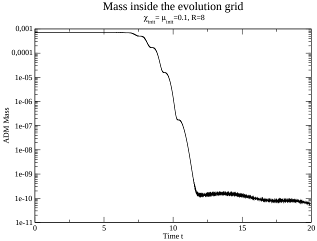
No compactification for hyperbolic equations

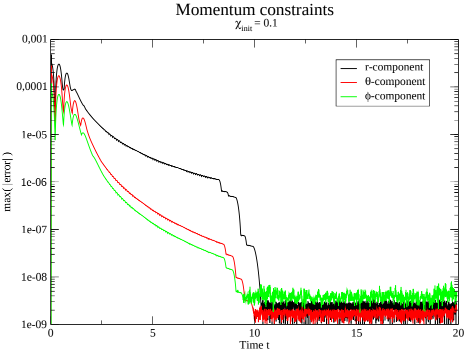
- boundary conditions at finite distance on a sphere of radius R
- perfectly absorbing for quadrupolar waves on Minkowski space-time (Novak & Bonazzola (2004))
- neglecting curvature effects (unlike Buchman & Sarbach 2007)
- reflection coefficient decays as δt^2 for $\ell \leq 2$ and R^{-4} for the others.

Constraints and ADM mass

Initial data: Gaussian profile with $\ell = m = 2$ component for $r^2 h^{rr}$ and $\mu(h^{ij})$ + conformal-thin sandwich approach (York 1999)

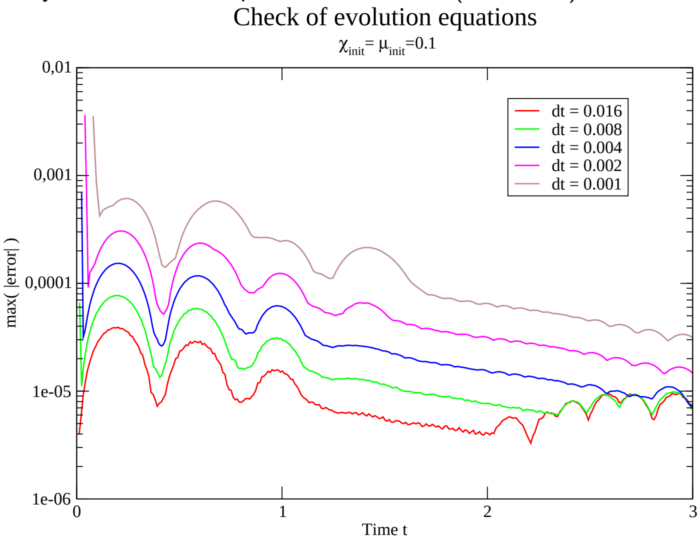






Evolution equations

Only two evolution equations are solved (out of six):



Summary and outlook

- New algorithm to solve the Einstein evolution equations, ensuring the gauge (divergence-free) condition, for any amplitude of the deviation from conformal-flatness,
- Solve only for two scalar wave equations,
- Designed for spectral methods in spherical coordinates (gain in CPU).

Next:

- Improve outgoing-wave boundary conditions,
- Implement the full linear operator (with the "shift advection"),
- Evolution of one black hole through the implementation of dynamical horizon boundary conditions,
- evolution with matter ("Mariage des Maillages").