Covariant conformal formulation of Einstein equations and Dirac formulation for numerical relativity

Eric Gourgoulhon and Jérôme Novak

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- Context
- 3+1 Formalism
- Choice and definitions of variables
- Einstein equations in terms of flat operators
- Maximal slicing and (generalized) Dirac gauge

Gravitational waves and numerical relativity

Compute by any means gravitational waveforms for interferometer data analysis (VIRGO, LIGO ...).

Simulate astrophysical systems in General Relativity

⇒ need of a formulation that can be implemented, is stable and can be applied to various physical systems.

3+1 formalism

Foliation of spacetime by spacelike hypersurfaces Σ_t

$$g_{\mu\nu} dx^\mu dx^\nu = -(N^2 - \beta_i \beta^i)\, dt^2 + 2\beta_i \, dt \, dx^i + \gamma_{ij} \, dx^i \, dx^j$$

Extrinsic curvature K_{ij} defined as: $\mathbf{K} = -\frac{1}{2} \pounds_{\mathbf{n}} \boldsymbol{\gamma}$.

- Hamiltonian constraint: $R + K^2 - K_{ij} K^{ij} = 16\pi E$
- Momentum constraint: $D_j K^{ij} - D^i K = 8\pi J^i$
- "Dynamical" equations:

$$\frac{\partial K_{ij}}{\partial t} - \pounds_{\boldsymbol{\beta}} K_{ij} = N \left[R_{ij} - 2K_{ik} K^k_{\ j} + K K_{ij} + 4\pi \left((S - E) - 2S_{ij} \right) \right] - D_i D_j N \, ,$$

$$\frac{\partial \gamma_{ij}}{\partial t} - \pounds_{\boldsymbol{\beta}} \gamma_{ij} = -2N K_{ij}$$

(R_{ij} : Ricci tensor of the 3-metric $\boldsymbol{\gamma}$, D_i : covariant derivative associated with $\boldsymbol{\gamma}$)

UNSTABLE for direct numerical integration

Change to variables $\tilde{\gamma}_{ij}$ and $\tilde{A}_{ij}$

Following *Smarr & York (1978)*, we define $\tilde{\gamma}_{ij} = \gamma^{-1/3}\gamma_{ij}$, which carries the dynamical degrees of freedom of γ_{ij} and is invariant under conformal transformation.

$\tilde{A}_{ij}$ is defined as $\tilde{A}_{ij} = \gamma^{-1/3}K_{ij} - \frac{1}{3}K\tilde{\gamma}_{ij}$

$\Rightarrow$ write Einstein equations in terms of these variables and "flat" operators (also define $\tilde{D}$, the slicing and gauge choices)

Tensor densities and flat metric

$\tilde{\gamma}_{ij}$ and $\tilde{A}_{ij}$ are not tensors but *tensor densities* of weight $-2/3$. For a tensor density $\boldsymbol{\tau}$ of weight n ($\tau = \gamma^{n/2}\, \boldsymbol{T}$), the covariant derivative can be defined as

$$\begin{aligned} D_k \tau^{i_1\cdots i_p}{}_{j_1\cdots j_q} = \quad & \partial_k \tau^{i_1\cdots i_p}{}_{j_1\cdots j_q} + \sum_{r=1}^{p} \Gamma^{i_r}_{kl} \tau^{i_1\cdots l\cdots i_p}{}_{j_1\cdots j_q} \\ & - \sum_{r=1}^{q} \Gamma^{l}_{kj_r} \tau^{i_1\cdots i_p}{}_{j_1\cdots l\cdots j_q} - \frac{n}{2} \partial_k \ln \gamma \tau^{i_1\cdots i_p}{}_{j_1\cdots j_q} \end{aligned}$$

Flat 3-metric f_{ij}, covariant derivative $\bar{D}_k$ and Christoffel symbols $\bar{\Gamma}^k_{ij}$.

$\bar{C}^k_{ij} = \Gamma^k_{ij} - \bar{\Gamma}^k_{ij} = \frac{1}{2}\gamma^{kl}\left[\bar{D}_i\gamma_{lj} + \bar{D}_j\gamma_{il} - \bar{D}_l\gamma_{ij}\right]$ is a tensor.

$\Delta^k_{ij} = \frac{1}{2}\tilde{\gamma}^{kl}\left[\bar{D}_i\tilde{\gamma}_{lj} + \bar{D}_j\tilde{\gamma}_{il} - \bar{D}_l\tilde{\gamma}_{ij}\right]$ is also a tensor and is used to define

$$\tilde{\Gamma}^k_{ij} = \Delta^k_{ij} + \bar{\Gamma}^k_{ij}$$

and therefore a covariant derivative $\tilde{D}_k$ such that $\tilde{D}_k\tilde{\gamma}_{ij} = 0$ (“associated” to $\tilde{\gamma}_{ij}$).

$$D_k \longrightarrow \tilde{D}_k \longrightarrow \bar{D}_k$$

Einstein equations using $\tilde{\gamma}_{ij}$, $\tilde{A}_{ij}$ and $\bar{D}_k$

$$\tilde{\gamma}^{kl}\bar{D}_k\bar{D}_l Q + H^k\bar{D}_k Q = \gamma^{1/6} N \left[\frac{1}{72}\tilde{\gamma}^{kl}\bar{D}_k \ln\gamma \bar{D}_l \ln\gamma + \frac{3}{2}\tilde{R}\right]$$

$$+\gamma^{1/2} N \left[4\pi S + \frac{3}{4}\tilde{A}_{kl}\tilde{A}^{kl} + \frac{K^2}{2}\right] - \gamma^{1/2}\left(\frac{\partial K}{\partial t} - \beta^k \bar{D}_k K\right) + \frac{\gamma^{1/6}}{6}\tilde{\gamma}^{kl}\bar{D}_k \ln\gamma \bar{D}_l N$$

$$\bar{D}_k\tilde{A}^{ik} + \Delta^i_{kl}\tilde{A}^{kl} + \frac{1}{2}\tilde{A}^{ik}\bar{D}_k \ln\gamma - \frac{2}{3}\tilde{\gamma}^{ik}\bar{D}_k K = 8\pi\gamma^{1/3} J^i$$

$$\frac{\partial}{\partial t}\ln\gamma = 2\bar{D}_k\beta^k + \beta^k\bar{D}_k \ln\gamma - 2NK$$

$$\frac{\partial\tilde{\gamma}^{ij}}{\partial t} = 2N\tilde{A}^{ij} - \tilde{\gamma}^{ik}\bar{D}_k\beta^j - \tilde{\gamma}^{jk}\bar{D}_k\beta^i + \frac{2}{3}\bar{D}_k\beta^k\tilde{\gamma}^{ij} + \beta^k\bar{D}_k\tilde{\gamma}^{ij}$$

$$\frac{\partial K}{\partial t} = -\gamma^{-1/3}\left[\tilde{\gamma}^{kl}\bar{D}_k\bar{D}_l N + H^k\bar{D}_k N + \frac{1}{6}\tilde{\gamma}^{kl}\bar{D}_k \ln\gamma \bar{D}_l N\right]$$

$$+N\left[4\pi(E+S) + \tilde{A}_{kl}\tilde{A}^{kl} + \frac{1}{3}K^2\right] + \beta^k\bar{D}_k K$$

with $Q = N\gamma^{1/6}$ and $H^i = \bar{D}_k\tilde{\gamma}^{ik}$.

$$\begin{aligned}
\frac{\partial \tilde{A}^{ij}}{\partial t} = \gamma^{-1/3} \Bigg\{ -\gamma^{1/6} \Bigg[& \tilde{\gamma}^{ik}\tilde{\gamma}^{jl}\bar{D}_k\bar{D}_l Q + \frac{1}{2}\left(\tilde{\gamma}^{il}\bar{D}_l\tilde{\gamma}^{jk} + \tilde{\gamma}^{jl}\bar{D}_l\tilde{\gamma}^{ik} - \tilde{\gamma}^{kl}\bar{D}_l\tilde{\gamma}^{ij}\right)\bar{D}_k Q \Bigg] \\
& + \frac{N}{2}\left[\tilde{\gamma}^{kl}\bar{D}_k\bar{D}_l\tilde{\gamma}^{ij} - \tilde{\gamma}^{ik}\bar{D}_k H^j + H^k\bar{D}_k\tilde{\gamma}^{ij} - \bar{D}_l\tilde{\gamma}^{ik}\bar{D}_k\tilde{\gamma}^{jl}\right. \\
& -\tilde{\gamma}_{kl}\tilde{\gamma}^{mr}\bar{D}_m\tilde{\gamma}^{ik}\bar{D}_r\tilde{\gamma}^{jl} + \tilde{\gamma}^{ik}\tilde{\gamma}_{ml}\bar{D}_k\tilde{\gamma}^{rm}\bar{D}_r\tilde{\gamma}^{jl} + \tilde{\gamma}^{jl}\tilde{\gamma}_{kr}\bar{D}_l\tilde{\gamma}^{mr}\bar{D}_m\tilde{\gamma}^{ik} \\
& \left. + \frac{1}{2}\tilde{\gamma}^{ik}\tilde{\gamma}^{jl}\bar{D}_k\tilde{\gamma}_{mr}\bar{D}_l\tilde{\gamma}^{mr} + \frac{1}{9}\tilde{\gamma}^{ik}\tilde{\gamma}^{jl}\bar{D}_k\ln\gamma\bar{D}_l\ln\gamma\right] \\
& + \frac{1}{3}\tilde{\gamma}^{ik}\tilde{\gamma}^{jl}\left(\bar{D}_k\ln\gamma\bar{D}_l N + \bar{D}_l\ln\gamma\bar{D}_k N\right) \\
& + \frac{1}{3}\Bigg[\gamma^{-1/6}\left(\tilde{\gamma}^{kl}\bar{D}_k\bar{D}_l Q + H^k\bar{D}_k Q\right) - \frac{2}{3}\tilde{\gamma}^{kl}\bar{D}_k\ln\gamma\bar{D}_l N \\
-N\left(-\bar{D}_k H^k + \frac{1}{4}\tilde{\gamma}^{kl}\bar{D}_k\tilde{\gamma}^{mr}\bar{D}_l\tilde{\gamma}_{mr} - \frac{1}{2}\tilde{\gamma}^{kl}\bar{D}_k\tilde{\gamma}^{mr}\bar{D}_r\tilde{\gamma}_{ml} + \frac{1}{18}\tilde{\gamma}^{kl}\bar{D}_k\ln\gamma\bar{D}_l\ln\gamma\right)\Bigg]\tilde{\gamma}^{ij}\Bigg\} \\
& + N\left[K\tilde{A}^{ij} + 2\tilde{\gamma}_{kl}\tilde{A}^{ik}\tilde{A}^{jl} - 8\pi\left(\gamma^{1/3}S^{ij} - \frac{1}{3}S\tilde{\gamma}^{ij}\right)\right] \\
& + \beta^k\bar{D}_k\tilde{A}^{ij} - \tilde{A}^{kj}\bar{D}_k\beta^i - \tilde{A}^{ik}\bar{D}_k\beta^j + \frac{2}{3}\bar{D}_k\beta^k\tilde{A}^{ij}
\end{aligned}$$

Maximal Slicing and Dirac Gauge

Maximal Slicing $(K = 0) \Rightarrow$

- singularity avoidance
- decoupling of elliptic equations

"Generalized" Dirac Gauge $(H^i = \bar{D}_k \tilde{\gamma}^{ki} = 0) \Rightarrow$

- transversality condition on γ^{ij} (separation of *coordinate* and *pure gravitational* waves)
- "scalar" d'Alembert operator for the evolution of $\tilde{\gamma}^{ij}$ (no second-order derivatives of $\tilde{\gamma}^{ij}$ in $\tilde{R}$)

During numerical evolution, this gauge can be imposed by infinitesimal coordinate change $x'^i = x^i + \xi^i$, ξ^i being determined by the elliptic equation

$$\tilde{\gamma}^{jk} \bar{D}_j \bar{D}_k \xi^i + \frac{1}{3} \tilde{\gamma}^{ij} \bar{D}_j \bar{D}_k \xi^k = H^i \left(1 + \frac{2}{3} \bar{D}_k \xi^k \right) + \xi^k \bar{D}_k H^i - H^k \bar{D}_k \xi^i$$

Summary and outlook

- covariant conformal decomposition of Einstein equations
- rigorous definition of $\tilde{D}$ “associated” to $\tilde{\gamma}_{ij}$
- arbitrary flat metric f_{ij} and definition of Dirac gauge in spherical coordinates

Future work:

- comparison with stationary models of rotating stars in MSQI gauge
- implementation for dynamical numerical integration