The Einstein equations on a computer: formulations and numerical solutions

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LPT Orsay, June, 2nd 2009

Relativistic gravity

In general relativity (1915), space-time is a four-dimensional Lorentzian manifold, where gravitational interaction is described by the metric $g_{\mu\nu}$.

Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$$

They form a set of 10 second-order non-linear PDEs, with very few (astro-)physically relevant exact solutions (Schwarzschild, Oppenheimer-Snyder, Kerr, ...).
$\Rightarrow$approximate solutions:
e.g. linearizing around the flat (Minkowski) solution in vacuum $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$:

$$\Box \left(h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \right) = -16\pi T_{\mu\nu}.$$

Using the linearized Einstein equations:

- at first order $h \sim \ddot{Q}$ (mass quadrupole momentum of the source), or further from the source $h \sim \frac{G}{c^4}\frac{E^{(\ell\geq 2)}}{r}$.
- the total gravitational power of a source is

$$L \sim \frac{G}{c^5} s^2 \omega^6 M^2 R^4.$$

...introducing the Schwarzschild radius $R_S = \dfrac{2GM}{c^2}$ and $\omega = v/r$:

$$L \sim \frac{c^5}{G} s^2 \left(\frac{R_S}{R}\right)^2 \left(\frac{v}{c}\right)^6$$

$\Rightarrow$non-spherical, relativistic compact objects:

- binary neutron stars or black holes,
- supernovae and neutron star oscillations.

Gravitational waves

Detectors

The effect of a wave on two tests-masses is the variation of their distance $\Delta l / l \sim h$, measured by a LASER beam.

LIGO: USA, Louisiana

LIGO: USA, Washington

VIRGO: France/Italy (Pisa)

Arms of these Michelson-type interferometers are 3 km (VIRGO) and 4 km (LIGO) long …almost perfect vacuum. They are acquiring data since 2005, with a very complex data analysis $\Rightarrow$need for accurate wave patterns: perturbative and numerical approaches.

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A brief history of numerical relativity

1966 : May & White, *Calculations of General-Relativistic Collapse*

1975 : Butterworth & Ipser, *Rapidly rotating fluid bodies in general relativity*

1976 : Smarr, Čadež, DeWitt & Eppley, *Collision of two black holes*

1985 : Stark & Piran, *Gravitational-Wave Emission from Rotating Gravitational Collapse*

1993 : Abrahams & Evans, *Vacuum axisymmetric gravitational collapse*

1999 : Shibata, *Fully general relativistic simulation of coalescing binary neutron stars*

2005 : Pretorius, *Evolution of Binary Black-Hole Spacetimes*

Formulations of Einstein equations

Four-dimensional approach

Classic approach in analytic studies: harmonic coordinate condition, the coordinates $\{x^\mu\}_{\mu=0\dots3}$ verify

$$\Box x^\mu = 0.$$

$\Rightarrow$nice form of Einstein equations, with $\Box g_{\alpha\beta} = S_{\alpha\beta}$,
$\Rightarrow$existence and uniqueness proofs in some cases.
However, the gauge can be pathological (e;g. in presence of matter): necessity of some generalization for numerical implementation.

$$\Box x^\mu = H^\mu,$$

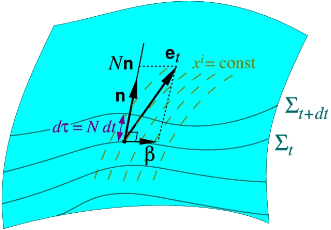
with an arbitrary source. Generalized Harmonic gauge
Choice of $H^\mu \iff$ choice of gauge

- arbitrary function,
- evolution toward harmonic gauge $\partial_t H_\mu = -\kappa(t) H_\mu$,
- prescription from 3+1 formulations (see later).

first successful simulation of binary black hole evolution

3+1 FORMALISM

Decomposition of spacetime and of Einstein equations



EVOLUTION EQUATIONS:

$$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}} K_{ij} =$$
$$-D_i D_j N + N R_{ij} - 2N K_{ik} K^k{}_j +$$
$$N \left[K K_{ij} + 4\pi ((S - E)\gamma_{ij} - 2S_{ij}) \right]$$
$$K^{ij} = \frac{1}{2N} \left(\frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i \right).$$

CONSTRAINT EQUATIONS:

$$R + K^2 - K_{ij} K^{ij} = 16\pi E,$$
$$D_j K^{ij} - D^i K = 8\pi J^i.$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -N^2\, dt^2 + \gamma_{ij} \left(dx^i + \beta^i dt \right) \left(dx^j + \beta^j dt \right)$$

Constrained / free formulations

As in electromagnetism, if the constraints are satisfied initially, they remain so for a solution of the evolution equations.

Free evolution

- start with initial data verifying the constraints,
- solve only the 6 evolution equations,
- recover a solution of all Einstein equations.

$\Rightarrow$apparition of constraint violating modes from round-off errors. Considered cures:

- Using of constraint damping terms and adapted gauges (many groups).
- Solving the constraints at every time-step (efficient elliptic solver?).

Fully-constrained formulation in Dirac gauge

Proposed by Bonazzola, Gourgoulhon, Grandclément & JN (2004): Define the conformal metric (carrying the dynamical degrees of freedom)

$$\tilde{\gamma}^{ij} = \Psi^4 \gamma^{ij} \text{ with } \Psi = \left(\frac{\det \gamma_{ij}}{\det f_{ij}} \right)^{1/12},$$

choose the generalized Dirac gauge

$$\nabla_j^{(f)} \tilde{\gamma}^{ij} = 0,$$

Then, one solves 4 constraint equations + 4 gauge equations (elliptic) at each time-step. Only 2 evolution equations.

Fully-constrained formulation

Properties of the hyperbolic part

The hyperbolic part is obtained combining the evolution equations:

$$\frac{\partial K_{ij}}{\partial t} - \boldsymbol{\mathcal{L}}_{\boldsymbol{\beta}} K_{ij} = \mathcal{S}_{ij} \text{ and } K^{ij} = \frac{1}{2N}\left(\frac{\partial \gamma^{ij}}{\partial t} + \dots\right),$$

to obtain a wave-type equation for $\tilde{\gamma}^{ij}$.

This system of evolution equations has been studied by Cordero-Carrión *et al.* (2008):

- the choice of Dirac gauge implies that the system is strongly hyperbolic
- can write it as conservation laws
- no incoming characteristic in the case of black hole excision technique

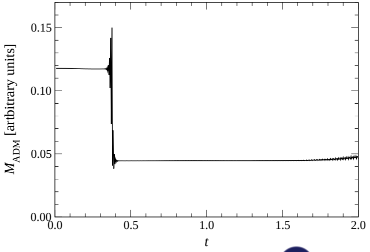
Elliptic part

Uniqueness issue

From the 4 constraints and the choice of time-slicing (gauge), an elliptic system of 5 non-linear equations can be formed

- Elliptic part of Einstein equations, to be solved at every time-step
- When setting $\tilde{\gamma}^{ij} = f^{ij}$, the system reduces to the Conformal-Flatness Condition (CFC).

Because of non-linear terms, the elliptic system may not converge $\Rightarrow$the case appears for dynamical, very compact matter and GW configurations (before appearance of the black hole).

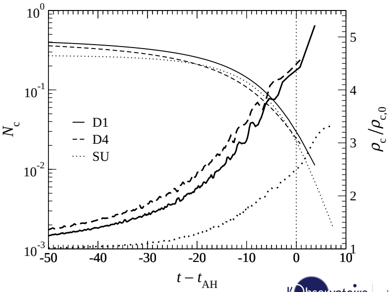


A solution to the uniqueness issue

Considering local uniqueness theorems for non-linear elliptic PDEs, it is possible to address the problem: $\Rightarrow$introducing auxiliary variables, to solve directly for the momentum constraints (Cordero-Carrión *et al.* (2009)

2nd fundamental form is rescaled by the conformal factor $A^{ij} = \Psi^{10} K^{ij}$, and decomposed into transverse and longitudinal parts $\Rightarrow$solving for each part:

- longitudinal $\Longleftrightarrow$ momentum constraint,
- transverse $\Longleftrightarrow$ zero (CFC) or evolution.



Summary of Einstein equations

constrained scheme

EVOLUTION

$$\frac{\partial A^{ij}}{\partial t} = \nabla^k \nabla_k \tilde{\gamma}^{ij} + \ldots$$

$$\frac{\partial \tilde{\gamma}^{ij}}{\partial t} = 2N\Psi^{-6}A^{ij} + \ldots$$

with

$$\det \tilde{\gamma}^{ij} = 1,$$

$$\nabla_j^{(f)} \tilde{\gamma}^{ij} = 0.$$

CONSTRAINTS

$$\begin{aligned} \nabla_j A^{ij} &= 8\pi\Psi^{10}S^i, \\ \Delta\Psi &= -2\pi\Psi^{-1}E \\ &\quad -\Psi^{-7}\frac{A^{ij}A_{ij}}{8}, \\ \Delta N\Psi &= 2\pi N\Psi^{-1} + \ldots \end{aligned}$$

with

$$\lim_{r\to\infty} \tilde{\gamma}^{ij} = f^{ij}, \lim_{r\to\infty} \Psi = \lim_{r\to\infty} N = 1.$$

Spectral methods for numerical relativity

Simplified picture
(see also Grandclément & JN 2009)

How to deal with functions on a computer?

$\Rightarrow$a computer can manage only integers

In order to represent a function $\phi(x)$ (e.g. interpolate), one can use:

- a finite set of its values $\{\phi_i\}_{i=0\ldots N}$ on a grid $\{x_i\}_{i=0\ldots N}$,
- a finite set of its coefficients in a functional basis $\phi(x) \simeq \sum_{i=0}^{N} c_i \Psi_i(x)$.

In order to manipulate a function (e.g. derive), each approach leads to:

- finite differences schemes

$$\phi'(x_i) \simeq \frac{\phi(x_{i+1}) - \phi(x_i)}{x_{i+1} - x_i}$$

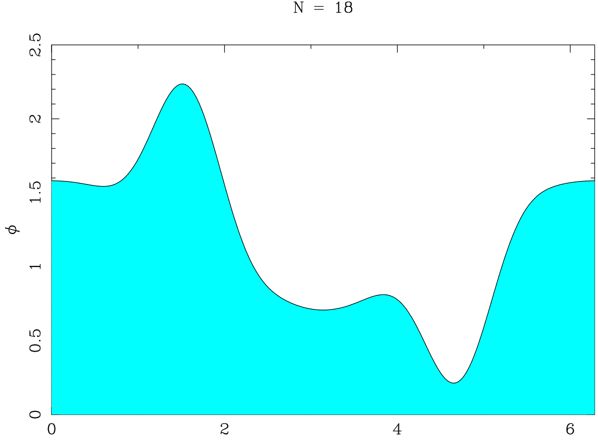
- spectral methods

$$\phi'(x) \simeq \sum_{i=0}^{N} c_i \Psi_i'(x)$$

Convergence of Fourier series

$$\phi(x) = \sqrt{1.5 + \cos(x)} + \sin^7 x$$

$$\phi(x) \simeq \sum_{i=0}^{N} a_i \Psi_i(x) \text{ with } \Psi_{2k} = \cos(kx),\ \Psi_{2k+1} = \sin(kx)$$



Use of orthogonal polynomials

The solutions $(\lambda_i, u_i)_{i\in\mathbb{N}}$ of a singular Sturm-Liouville problem on the interval $x \in [-1, 1]$:

$$-\left(pu'\right)' + qu = \lambda wu,$$

with $p > 0, \mathcal{C}^1, p(\pm 1) = 0$

- are orthogonal with respect to the measure w:

$$(u_i, u_j) = \int_{-1}^{1} u_i(x) u_j(x) w(x) \mathrm{d}x = 0 \text{ for } m \neq n,$$

- form a spectral basis such that, if $f(x)$ is smooth ($\mathcal{C}^\infty$)

$$f(x) \simeq \sum_{i=0}^{N} c_i u_i(x)$$

  converges faster than any power of N (usually as e^{-N}).

Gauss quadrature to compute the integrals giving the c_i's.
Chebyshev, Legendre and, more generally any type of Jacobi polynomial enters this category.

General form of an ODE of unknown $u(x)$:

$$\forall x \in [a,b],\ Lu(x) = s(x),\ \text{and}\ \ Bu(x)|_{x=a,b} = 0,$$

The approximate solution is sought in the form

$$\bar{u}(x) = \sum_{i=0}^{N} c_i \Psi_i(x).$$

The $\{\Psi_i\}_{i=0\dots N}$ are called trial functions: they belong to a finite-dimension sub-space of some Hilbert space $\mathcal{H}_{[a,b]}$.
$\bar{u}$ is said to be a numerical solution if:

- $B\bar{u} = 0$ for $x = a, b$,
- $R\bar{u} = L\bar{u} - s$ is "small".

Defining a set of test functions $\{\xi_i\}_{i=0\dots N}$ and a scalar product on $\mathcal{H}_{[a,b]}$, R is small iff:

$$\forall i = 0 \dots N, \quad (\xi_i, R) = 0.$$

$\boxed{\text{It is expected that } \lim_{N\to\infty} \bar{u} = u, \text{ “true” solution of the ODE.}}$

Various numerical methods

type of trial functions Ψ

- finite-differences methods for local, overlapping polynomials of low order,
- finite-elements methods for local, smooth functions, which are non-zero only on a sub-domain of $[a, b]$,
- spectral methods for global smooth functions on $[a, b]$.

type of test functions ξ for spectral methods

- tau method: $\xi_i(x) = \Psi_i(x)$, but some of the test conditions are replaced by the boundary conditions.
- collocation method (pseudospectral): $\xi_i(x) = \delta(x - x_i)$, at collocation points. Some of the test conditions are replaced by the boundary conditions.
- Galerkin method: the test and trial functions are chosen to fulfill the boundary conditions.

Inversion of linear ODEs

Thanks to the well-known recurrence relations of Legendre and Chebyshev polynomials, it is possible to express the coefficients $\{b_i\}_{i=0\dots N}$ of

$$Lu(x) = \sum_{i=0}^{N} b_i \left| \begin{array}{l} P_i(x) \\ T_i(x) \end{array} \right., \text{ with } u(x) = \sum_{i=0}^{N} a_i \left| \begin{array}{l} P_i(x) \\ T_i(x) \end{array} \right. .$$

If $L = \mathrm{d}/\mathrm{d}x, x\times, \dots,$ and $u(x)$ is represented by the vector $\{a_i\}_{i=0\dots N}$, L can be approximated by a matrix.

Resolution of a linear ODE

$$\Updownarrow$$

inversion of an $(N+1) \times (N+1)$ matrix

With non-trivial ODE kernels, one must add the boundary conditions to the matrix to make it invertible!

Some singular operators

$$\boxed{u(x) \mapsto \frac{u(x)}{x} \text{ is a linear operator, inverse of } u(x) \mapsto xu(x).}$$

Its action on the coefficients $\{a_i\}_{i=0\dots N}$ representing the N-order approximation to a function $u(x)$ can be computed as the product by a regular matrix.⇒The computation in the coefficient space of $u(x)/x$, on the interval $[-1, 1]$ always gives a finite result (both with Chebyshev and Legendre polynomials).

⇒The actual operator which is thus computed is

$$u(x) \mapsto \frac{u(x) - u(0)}{x}.$$

⇒Compute operators in spherical coordinates, with coordinate singularities

$$\text{e.g. } \Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\theta\varphi}$$

Time discretization

Formally, the representation (and manipulation) of $f(t)$ is the same as that of $f(x)$.
$\Rightarrow$in principle, one should be able to represent a function $u(x,t)$ and solve time-dependent PDEs only using spectral methods...but this is not the way it is done! Two works:

- Ierley *et al.* (1992): study of the Korteweg de Vries and Burger equations, Fourier in space and Chebyshev in time $\Rightarrow$time-stepping restriction.
- Hennig and Ansorg (2008): study of non-linear (1+1) wave equation, with conformal compactification in Minkowski space-time. $\Rightarrow$nice spectral convergence.

WHY?

- poor *a priori* knowledge of the exact time interval,
- too big matrices for full 3+1 operators ($\sim 30^4 \times 30^4$),
- finite-differences time-stepping errors can be quite small.

Explicit / implicit schemes

Let us look for the numerical solution of (L acts only on x):

$$\forall t \geq 0, \quad \forall x \in [-1, 1], \quad \frac{\partial u(x,t)}{\partial t} = Lu(x,t),$$

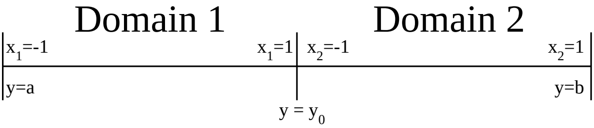
with good boundary conditions. Then, with δt the time-step: $\forall J \in \mathbb{N}, \quad u^J(x) = u(x, J \times \delta t)$, it is possible to discretize the PDE as

- $u^{J+1}(x) = u^J(x) + \delta t\, Lu^J(x)$: explicit time scheme (forward Euler); easy to implement, fast but limited by the CFL condition.
- $u^{J+1}(x) - \delta t\, Lu^{J+1}(x) = u^J(x)$: implicit time scheme (backward Euler); one must solve an equation (ODE) to get u^{J+1}, the matrix approximating it here is $I - \delta t\, L$. Allows longer time-steps but slower and limited to second-order schemes.

Multi-domain approach

Multi-domain technique : several touching, or overlapping, domains (intervals), each one mapped on $[-1, 1]$.

- boundary between two domains can be the place of a discontinuity $\Rightarrow$recover spectral convergence,
- one can set a domain with more coefficients (collocation points) in a region where much resolution is needed $\Rightarrow$fixed mesh refinement,
- 2D or 3D, allows to build a complex domain from several simpler ones,



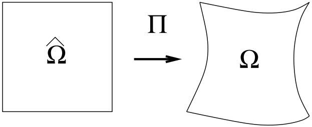
Depending on the PDE, matching conditions are imposed at $y = y_0 \iff$ boundary conditions in each domain.

Mappings and multi-D

In two spatial dimensions, the usual technique is to write a function as:

$$f \quad : \quad \hat{\Omega} = [-1,1] \times [-1,1] \to \mathbb{R}$$

$$f(x,y) \quad = \quad \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} c_{ij} P_i(x) P_j(y)$$



The domain $\hat{\Omega}$ is then mapped to the real physical domain, trough some mapping $\Pi : (x,y) \mapsto (X,Y) \in \Omega$.

$\Rightarrow$When computing derivatives, the Jacobian of Π is used.

COMPACTIFICATION

A very convenient mapping in spherical coordinates is

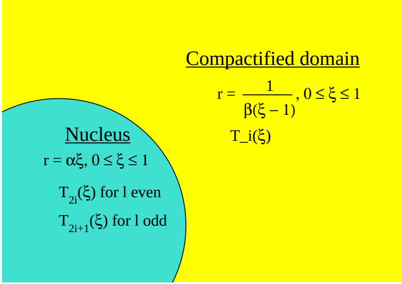
$$x \in [-1,1] \mapsto r = \frac{1}{\alpha(x-1)},$$

to impose boundary condition for $r \to \infty$ at $x = 1$.

Example:

3D Poisson equation, with non-compact support

To solve $\Delta\phi(r,\theta,\varphi) = s(r,\theta,\varphi)$, with s extending to infinity.



- setup two domains in the radial direction: one to deal with the singularity at $r = 0$, the other with a compactified mapping.
- In each domain decompose the angular part of both fields onto spherical harmonics:

$$\phi(\xi,\theta,\varphi) \simeq \sum_{\ell=0}^{\ell_{\max}} \sum_{m=-\ell}^{m=\ell} \phi_{\ell m}(\xi) Y_\ell^m(\theta,\varphi),$$

- $\forall(\ell, m)$ solve the ODE: $\dfrac{\mathrm{d}^2\phi_{\ell m}}{\mathrm{d}\xi^2} + \dfrac{2}{\xi}\dfrac{\mathrm{d}\phi_{\ell m}}{\mathrm{d}\xi} - \dfrac{\ell(\ell+1)\phi_{\ell m}}{\xi^2} = s_{\ell m}(\xi),$
- match between domains, with regularity conditions at $r = 0$, and boundary conditions at $r \to \infty$.

Numerical simulation of black holes

Puncture methods

... *it is not yet clear how and why they work.* Hannam *et al.* (2007)

- black holes are described in the initial data in coordinates that do not reach the physical singularity,

$\Rightarrow$ the coordinates follow a wormhole through another copy of the asymptotically flat exterior spacetime,

- this is compactified so that infinity is represented by a single point, called "puncture".

$$\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij} \text{ with } \Psi \sim \frac{1}{r}, \text{ use of } \phi = \log \Psi \text{ or } \chi = \Psi^{-4}.$$

BUT

During the evolution the time-slice loses contact with the second asymptotically flat end, and finishes on a cylinder of finite radius.

$$\Psi(t=0) = \mathcal{O}\left(\frac{1}{r}\right) \text{ evolves into } \Psi(t>0) = \mathcal{O}\left(\frac{1}{\sqrt{r}}\right).$$

Use of the shift vector β^i to generate motion.

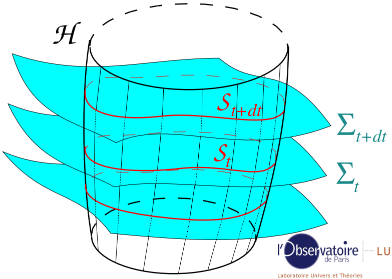
Excision techniques

apparent horizons as a boundary

- Remove a neighborhood of the central singularity from computational domain;
- Replace it with boundary conditions on this newly obtained boundary (usually, a sphere),
- Until now, imposition of apparent horizon / isolated horizon properties: zero expansion of outgoing light rays.

⇒New views on the concept of black hole, following works by Hayward, Ashtekar and Krishnan:

- Quasi-local approach, making the black hole a causal object;
- For hydrodynamic, electromagnetic and gravitational waves (Dirac gauge): no incoming characteristics.



Excision technique

Kerr solution from boundary conditions

Can one recover a Kerr black hole only from boundary conditions and Einstein equations?
⇒Many computations with CFC, but there is no time slicing in which (the spatial part of) Kerr solution can be conformally flat (Garat & Price 2000).
Vasset, JN & Jaramillo (2009) recover full Kerr solution

- constant value (N), zero expansion on the horizon (ψ);
- rotation state for β^θ, β^ϕ and isolated horizon for β^r;
- NO condition for $\tilde{\gamma}^{ij}$;

\+ asymptotic flatness and Einstein equations!

In particular, no symmetry requirement has been imposed in the "bulk" (only on the horizon) ⇒illustration of the rigidity theorem by Hawking & Ellis (1973).

Summary - Perspectives

- Many new results in numerical relativity,
- The Fully-constrained Formulation is needed for long-term evolutions, particularly in the cases of gravitational collapse,
- This formulation is now well-studied and stable.

Many of the numerical features presented here are available in the LORENE library: `http://lorene.obspm.fr`, publicly available under GPL.

Future directions:

- Implementation of FCF and excision methods in the collapse code to simulate the formation of a black hole;
- Use of excision techniques in the dynamical case

$\Rightarrow$most of groups are now heading toward more complex physics: electromagnetic field, realistic equation of state for matter, ...

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