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Solution of the Gravitational Wave Tensor Equation Using Spectral Methods

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based on collaboration with
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Outline

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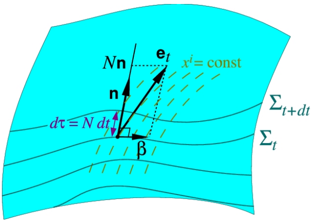
3+1 FORMALISM

Decomposition of spacetime and of Einstein equations



EVOLUTION EQUATIONS:

$$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\beta} K_{ij} =$$
$$-D_i D_j N + N R_{ij} - 2N K_{ik} K^k_{\ j} +$$
$$N \left[K K_{ij} + 4\pi ((S - E)\gamma_{ij} - 2S_{ij}) \right]$$
$$K^{ij} = \frac{1}{2N} \left(\frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i \right).$$

CONSTRAINT EQUATIONS:

$$R + K^2 - K_{ij} K^{ij} = 16\pi E,$$
$$D_j K^{ij} - D^i K = 8\pi J^i.$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -N^2\, dt^2 + \gamma_{ij} \left(dx^i + \beta^i dt \right) \left(dx^j + \beta^j dt \right)$$

Flat metric and Dirac gauge

Following Bonazzola *et al.* (2004)

We introduce f_{ij} (with $\frac{\partial f_{ij}}{\partial t} = 0$) as the asymptotic structure of γ_{ij}, and $\mathcal{D}_i$ the associated covariant derivative.

Define:

$$\tilde{\gamma}_{ij} := \Psi^{-4}\,\gamma_{ij} \text{ or } \gamma_{ij} =: \Psi^4\,\tilde{\gamma}_{ij} \quad \text{with} \quad \Psi := \left(\frac{\gamma}{f}\right)^{1/12}$$

$\tilde{\gamma}_{ij}$ is invariant under any conformal transformation of γ_{ij} and verifies $\det \tilde{\gamma}_{ij} = f$

Finally,

$$\tilde{\gamma}^{ij} = f^{ij} + h^{ij}$$

is the deviation of the 3-metric from conformal flatness.

Generalization the gauge introduced by Dirac (1959) to any type of coordinates:

divergence-free condition on $\tilde{\gamma}^{ij}$

$$\mathcal{D}_j \tilde{\gamma}^{ij} = \mathcal{D}_j h^{ij} = 0$$

Einstein equations

Dirac gauge and maximal slicing ($K = 0$)

Constraint Equations

$$\begin{aligned} \Delta\Psi &= \mathcal{S}_{\mathrm{Ham}}, \\ \Delta\beta^i + \frac{1}{3}\mathcal{D}^i\left(\mathcal{D}_j\beta^j\right) &= \mathcal{S}_{\mathrm{Mom}}. \end{aligned}$$

Trace of dynamical equations

$$\Delta N = \mathcal{S}_{\dot{K}}$$

Evolution equations

$$\frac{\partial^2 h^{ij}}{\partial t^2} - \frac{N^2}{\Psi^4}\Delta h^{ij} - 2\pounds_{\boldsymbol{\beta}}\frac{\partial h^{ij}}{\partial t} + \pounds_{\boldsymbol{\beta}}\pounds_{\boldsymbol{\beta}} h^{ij} = \mathcal{S}^{ij}_{\mathrm{Dyn}}$$

Evolution Equation

Position of the problem

- Wave-like equation for a symmetric tensor:
  6 components - 3 Dirac gauge conditions - $\left(\det \tilde{\gamma}^{ij} = 1\right)$
  $\Rightarrow$2 degrees of freedom
- Work with $h = f_{ij} h^{ij} = 0$ (for the moment): asymptotically equivalent to $\left(\det \tilde{\gamma}^{ij} = 1\right)$ - non-linear condition;
- the evolution operator appearing is not, in general, hyperbolic (complex eigenvalues); with the Dirac gauge, it is (result by I. Cordero).

Simplified numerical problem:

- solve a flat wave equation for a symmetric tensor $\Box h^{ij} = \mathcal{S}^{ij}$,
- ensure the gauge condition $\mathcal{D}_j h^{ij} = 0$,
- has a given value of the trace.

Outgoing boundary conditions

- If no compactification is done, it is necessary to impose boundary condition at a finite distance R;
- Far enough from the source, one can consider the evolution operator as being a flat Dalembert operator;
- It is then possible to use outgoing-wave boundary condition.

BUT

- Usual outgoing-wave condition (Sommerfeld) is exact, up to numerical scheme precision, only for $\ell = 0$ mode.

$\Rightarrow$Use of enhanced condition (Novak & Bonazzola (2004)):

- exact (up to discretization error) $\forall \ell \leq 2$,
- for $\ell > 2$, the reflected wave decreases as $1/R^4$ (versus $1/R^2$ for Sommerfeld).

Boundary conditions at a black hole horizon

Under development...

- Use of excision technique for black hole evolution ⇒at the *apparent horizon* (See talk by E. Gourgoulhon);
- In this region, the evolution operator for h^{ij} must be taken with all (linear) terms,

Then, in the Dirac gauge, for a dynamical horizon:

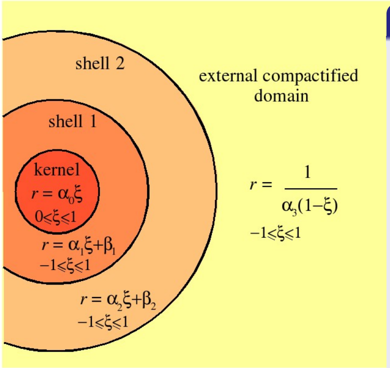
- All characteristics are outgoing...
- ... no boundary condition must be imposed.

Study by I. Cordero

OK with the intuition of a spacelike boundary of the computational domain.

Multidomain 3D decomposition

numerical library Lorene (http://www.lorene.obspm.fr)



Decomposition:

Chebyshev polynomials for ξ, Fourier or Y_ℓ^m for the angular part (θ, ϕ),

- symmetries and regularity conditions of the fields at the origin and on the axis of spherical coordinate system
- compactified variable for elliptic PDEs $\Rightarrow$boundary conditions are well imposed

Drawback: Gibbs phenomenon!

Solutions of Poisson and wave equations

The angular part of any field ϕ is decomposed on a set of spherical harmonics $Y_\ell^m(\theta,\varphi)$, which are eigenvectors of the angular part of the Laplace operator

$$\Delta_{\theta\varphi} Y_\ell^m = -\ell(\ell+1) Y_\ell^m$$

$\Delta\phi = \sigma$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} \right) \phi_{\ell m}(r) = \sigma_{\ell m}(r)$$

Accuracy on the solution $\sim 10^{-13}$ (exponential decay)

$\Box\phi = \sigma$

$$\left[1 - \frac{\delta t^2}{2} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} \right) \right] \phi_{\ell m}^{J+1} = \sigma_{\ell m}^J$$

Accuracy on the solution $\sim 10^{-10}$ (time-differencing)

$\forall(\ell,m)$ the operator inversion $\iff$ inversion of a $\sim 30 \times 30$ matrix
Non-linear parts are evaluated in the physical space and contribute as sources to the equations.

Spherical coordinates and components

Choice for f_{ij} : spherical polar coordinates

- stars and black holes are of spheroidal shape
- compactification made easy (only r)
- use of spherical harmonics
- grid boundaries are smooth surfaces

Use of spherical orthonormal triad (tensor components)

- Dirac gauge can easily be imposed
- asymptotically, it is easier to extract gravitational waves

Vector spherical harmonics

Following *e.g.* Thorne (1980)

A 3D vector field $\boldsymbol{V}$ can be decomposed onto a set of vector spherical harmonics

$$\boldsymbol{V} = \sum_{\ell,m} R_{\ell m}(r)\boldsymbol{Y}^R_{\ell m}(\theta,\varphi) + E_{\ell m}(r)\boldsymbol{Y}^E_{\ell m}(\theta,\varphi) + B_{\ell m}(r)\boldsymbol{Y}^B_{\ell m}(\theta,\varphi),$$

- pure spin vector harmonics,
- orthonormal set of regular angular functions,
- not eigenfunctions of vector angular Laplacian

$$\begin{aligned} \boldsymbol{Y}^R_{\ell m} &\propto Y_{\ell m}\boldsymbol{r}, \ \text{(longitudinal)} \\ \boldsymbol{Y}^E_{\ell m} &\propto \boldsymbol{\mathcal{D}}Y_{\ell m}, \ \text{(transverse)} \\ \boldsymbol{Y}^B_{\ell m} &\propto \boldsymbol{r}\times\boldsymbol{\mathcal{D}}Y_{\ell m} \ \text{(transverse)} \end{aligned}$$

$V^r = \sum R_{\ell m}(r)Y_{\ell m}(\theta,\varphi)$, and we define two other potentials

$$\begin{aligned} V^\theta &= \frac{\partial\eta}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial\mu}{\partial\varphi}, \\ V^\varphi &= \frac{1}{\sin\theta}\frac{\partial\eta}{\partial\varphi} + \frac{\partial\mu}{\partial\theta}; \end{aligned}$$

$$\begin{aligned} \eta(r,\theta,\varphi) &= \sum_{\ell,m} E_{\ell m}(r)Y_{\ell m}, \\ \mu(r,\theta,\varphi) &= \sum_{\ell,m} B_{\ell m}(r)Y_{\ell m} \end{aligned}$$

Differential operators in terms of new potentials

Flat wave operator $\Box V^i = S^i$ (divergence-free case)

$$
\begin{aligned}
-\frac{\partial^2 V^r}{\partial t^2} + \Delta V^r + \frac{2}{r}\frac{\partial V^r}{\partial r} + \frac{2V^r}{r^2} &= S^r, \\
-\frac{\partial^2 \eta}{\partial t^2} + \Delta \eta + \frac{2}{r}\frac{\partial V^r}{\partial r} &= \eta_S, \\
-\frac{\partial^2 \mu}{\partial t^2} + \Delta \mu &= \mu_S.
\end{aligned}
$$

Divergence-free condition $\mathcal{D}_i V^i = 0$

$$
\frac{\partial V^r}{\partial r} + \frac{2V^r}{r} + \frac{1}{r}\Delta_{\theta\varphi}\eta = 0
$$

... thus μ does not depend on the divergence of $\boldsymbol{V}$.

Helmholtz decomposition

Any vector field $\boldsymbol{V}$ on $\mathbb{R}^3$, twice continuously differentiable and with rapid enough decay at infinity can be uniquely written as

$$\boldsymbol{V} = \tilde{\boldsymbol{V}} + \boldsymbol{\mathcal{D}}\phi, \text{ with } \mathcal{D}_i \tilde{V}^i = 0.$$

from $\boldsymbol{\mathcal{D}} \times \boldsymbol{V} = \boldsymbol{\mathcal{D}} \times \tilde{\boldsymbol{V}}$, one gets

$$\mu_V = \mu_{\tilde{V}} \text{ (twice: } r\text{- and } \eta\text{- components)} ,$$

$$\frac{\partial \eta_V}{\partial r} + \frac{\eta_V}{r} - \frac{V^r}{r} = \frac{\partial \eta_{\tilde{V}}}{\partial r} + \frac{\eta_{\tilde{V}}}{r} - \frac{\tilde{V}^r}{r} \text{ (}\mu\text{- component)} .$$

⇒the quantities

$$A = \frac{\partial \eta}{\partial r} + \frac{\eta}{r} - \frac{V^r}{r}$$

and μ are not sensitive to the gradient part of a vector.

Evolution equations

ensuring divergence-free condition...

From the definition of A and the expression of the wave operator for a vector, one gets for the source ($\Box V^i = S^i$)

$$A_S = \frac{\partial \eta_S}{\partial r} + \frac{\eta_S}{r} - \frac{S^r}{r},$$

and

$$\Box A_V = A_S$$

once A is known, one can reconstruct the vector V^i from

$$\begin{aligned} \frac{\partial \eta}{\partial r} + \frac{\eta}{r} - \frac{V^r}{r} &= A, \\ \frac{\partial V^r}{\partial r} + \frac{2V^r}{r} + \frac{1}{r}\Delta_{\theta\varphi}\eta &= 0 \text{ divergence-free condition.} \end{aligned}$$

and μ (since $\Box \mu = \mu_S$).

Integration procedure

1. from S^i compute A_S and μ_S,
2. solve the equation for μ,
3. solve the equation for A,
4. solve the coupled system given by the divergence-free condition and the definition of A to get V^r and η,
5. reconstruct V^i from V^r, η and μ.

Tensor spherical harmonics

A 3D symmetric tensor field $\boldsymbol{h}$ can be decomposed onto a set of tensor pure spin spherical harmonics and one can get 6 scalar potentials to represent the tensor:

$\boldsymbol{T}^{L_0}$	$\boldsymbol{T}^{T_0}$	$\boldsymbol{T}^{E_1}$	$\boldsymbol{T}^{B_1}$	$\boldsymbol{T}^{E_2}$	$\boldsymbol{T}^{B_2}$
h^{rr}	$\tau = h^{\theta\theta} + h^{\varphi\varphi}$	η	μ	W	X

with the following relations:

$$
\begin{aligned}
h^{r\theta} &= \frac{\partial \eta}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial \mu}{\partial \varphi}, \\
h^{r\varphi} &= \frac{1}{\sin\theta}\frac{\partial \eta}{\partial \varphi} + \frac{\partial \mu}{\partial \theta}, \\
\frac{h^{\theta\theta} - h^{\varphi\varphi}}{2} &= \frac{\partial^2 W}{\partial \theta^2} - \frac{1}{\tan\theta}\frac{\partial W}{\partial \theta} - \frac{1}{\sin^2\theta}\frac{\partial^2 W}{\partial \varphi^2} - 2\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial X}{\partial \varphi}\right), \\
h^{\theta\varphi} &= \frac{\partial^2 X}{\partial \theta^2} - \frac{1}{\tan\theta}\frac{\partial X}{\partial \theta} - \frac{1}{\sin^2\theta}\frac{\partial^2 X}{\partial \varphi^2} + 2\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial W}{\partial \varphi}\right).
\end{aligned}
$$

Differential operators

Divergence-free condition $H^i = \mathcal{D}_j h^{ij} = 0$

$$
\begin{aligned}
H^r &= \frac{\partial h^{rr}}{\partial r} + \frac{2h^{rr}}{r} + \frac{1}{r}\Delta_{\theta\varphi}\eta - \frac{\tau}{r} = 0,\\
H^\eta &= \frac{\partial \eta}{\partial r} + \frac{3\eta}{r} + \left(\Delta_{\theta\varphi} + 2\right)\frac{W}{r} + \frac{\tau}{2r} = 0,\\
H^\mu &= \frac{\partial \mu}{\partial r} + \frac{3\mu}{r} + \left(\Delta_{\theta\varphi} + 2\right) X = 0;
\end{aligned}
$$

"Electric type" potentials	"Magnetic type"
h^{rr}, τ, η, W	μ, X

$\Rightarrow$two groups of coupled equations for the wave operator.

Divergence-free part of a symmetric tensor

As for the Helmholtz decomposition:

$$h^{ij} = \tilde{h}^{ij} + \mathcal{D}^i V^j + \mathcal{D}^j V^i$$

... but no possibility to use the curl operator on a symmetric tensor!

3 degrees of freedom for $\tilde{h}$

$$\begin{aligned}
A &= \frac{\partial X}{\partial r} - \frac{\mu}{r}, \\
B &= \frac{\partial W}{\partial r} - \frac{1}{2r}\Delta_{\theta\varphi} W - \frac{\eta}{r} + \frac{\tau}{4r}, \\
C &= \frac{\partial \tau}{\partial r} - \frac{2h^{rr}}{r} - 2\Delta_{\theta\varphi}\left(\frac{\partial W}{\partial r} + \frac{W}{r}\right)
\end{aligned}$$

Wave equation $\Box h^{ij} = S^{ij}$

$$\begin{aligned}
\Box A &= A_S, \\
\Box B + \frac{C}{2r^2} &= B_S, \\
\Box C - \frac{2C}{r^2} - \frac{8\Delta_{\theta\varphi} B}{r^2} &= C_S.
\end{aligned}$$

Divergence-free evolution

Define ℓ by ℓ

$$\tilde{B}_{\ell m} = 2B_{\ell m} + \frac{C_{\ell m}}{2(\ell+1)},$$
$$\tilde{C}_{\ell m} = 2B_{\ell m} - \frac{C_{\ell m}}{2\ell};$$

Wave equation $\Box h^{ij} = S^{ij}$

$$\Box \tilde{B} + \frac{2\ell \tilde{B}}{r^2} = \tilde{B}_S,$$
$$\Box \tilde{C} - \frac{2(\ell+1)\tilde{C}}{r^2} = \tilde{C}_S.$$

In the case where $f_{ij}h^{ij} = 0$ ($h^{rr} = -\tau$):

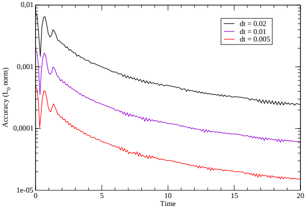
1. compute A_S and $\tilde{B}_S$,
2. solve wave equations for A and $\tilde{B}$ (a wave operator shifted in ℓ),
3. solve the system composed of
   - definition of A
   - $H^\mu = 0$ (Dirac gauge)

   on the one hand, and

   - definition of $\tilde{B}$
   - $H^r = 0$
   - $H^\eta = 0$

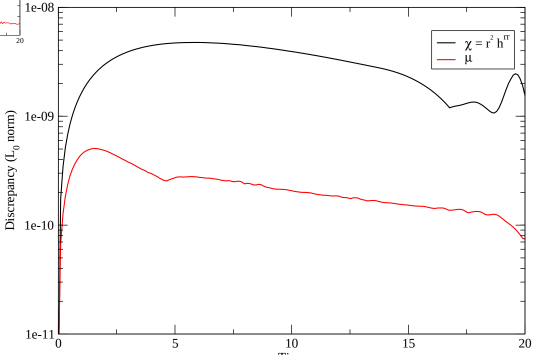
   on the other hand,
4. recover the tensor components.

Numerical Tests
Is the wave equation solved?

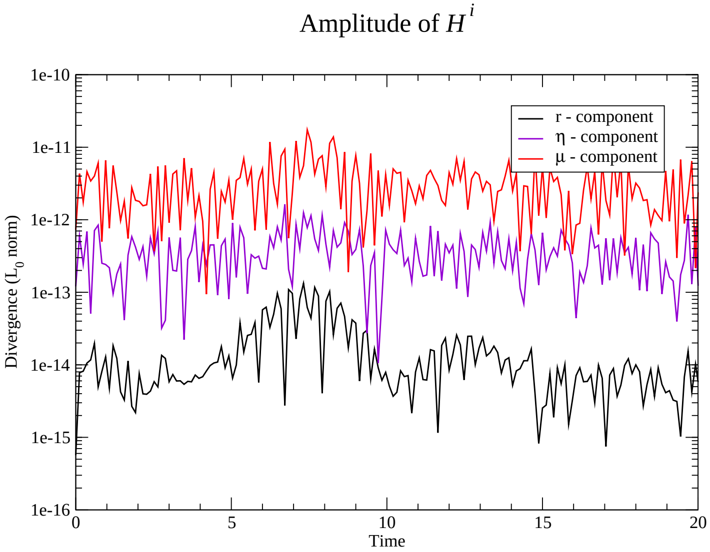


Initial data: Gaussian profile for h^{rr} and μ, with $\ell = 2$ and $\ell = 3$ modes. Evolution compared to the method of Bonazzola *et al.* (2004)

$\Box h^{ij} = 0$, with $dt = 0.02$, $R = 20$. 4 domains with 33 points in each.

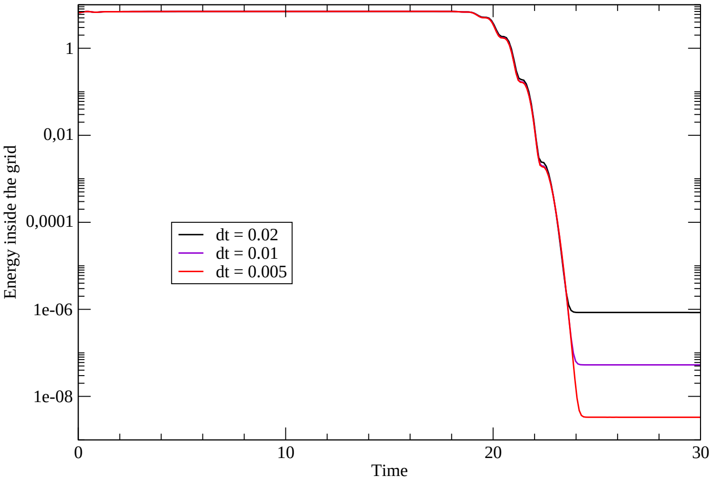


Numerical Tests
Is the solution divergence-free?



Numerical Tests

Are the boundary conditions still transparent?



Summary and outlook

- Algorithm to solve the tensor wave equation, ensuring the divergence-free condition,
- In the traceless case, solve only for two scalar wave equations,
- Designed for spectral methods in spherical coordinates (gain in CPU).
- Test it with the full Einstein equations,
- Take into account the full linear operator (with the "shift advection"),
- Evolution of one black hole,
- Extension to bi-spherical coordinates (Ansorg 2005)...

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Inversion formulas

$$
\begin{aligned}
\Delta_{\theta\varphi}\eta &= \left(\frac{\partial h^{r\theta}}{\partial\theta} + \frac{h^{r\theta}}{\tan\theta} + \frac{1}{\sin\theta}\frac{\partial h^{r\varphi}}{\partial\varphi}\right)\\
\Delta_{\theta\varphi}\mu &= \left(\frac{\partial h^{r\varphi}}{\partial\theta} + \frac{h^{r\varphi}}{\tan\theta} - \frac{1}{\sin\theta}\frac{\partial h^{r\theta}}{\partial\varphi}\right),\\
\Delta_{\theta\varphi}\left(\Delta_{\theta\varphi}+2\right)W &= \frac{\partial^2 P}{\partial\theta^2} + \frac{3}{\tan\theta}\frac{\partial P}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2 P}{\partial\varphi^2} - 2P\\
&+ \frac{2}{\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{\partial h^{\theta\varphi}}{\partial\theta} + \frac{h^{\theta\varphi}}{\tan\theta}\right),\\
\Delta_{\theta\varphi}\left(\Delta_{\theta\varphi}+2\right)X &= \frac{\partial^2 h^{\theta\varphi}}{\partial\theta^2} + \frac{3}{\tan\theta}\frac{\partial h^{\theta\varphi}}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2 h^{\theta\varphi}}{\partial\varphi^2} - 2h^{\theta\varphi}\\
&- \frac{2}{\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{\partial P}{\partial\theta} + \frac{P}{\tan\theta}\right).
\end{aligned}
$$